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Solution of coupled Whitham–Broer–Kaup equations using optimal homotopy asymptotic method

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ABSTRACT

In this work, an approximate method called Optimal Homotopy Asymptotic Method (OHAM) is proposed and implemented for the numerical solution of Whitham–Broer–Kaup (WBK) equations with blow-up and periodic solutions. Results obtained through OHAM are compared with the exact as well as with the results available in the literature. It was revealed that only second-order OHAM solutions are sufficient to achieve the desired accuracy in comparison to the higher order solutions of the methods we have made comparison with.

1. Introduction

Most of the phenomena of physical and engineering interest are governed by nonlinear differential equations (DEs). So, for the last few decades, a great deal of attention has been directed towards the solution (both exact and numerical) of these problems. Unfortunately, very few of them have their exact solution and the researchers are forced to look for the numerical solution of such problems. Various methods are available in the literature for the exact and numerical solution of these problems. These include the homogeneous balance method, Cole–Hopf transformation, tanh method, sine–cosine method, similarity reduction method, Darboux transformation, Backland transformation (Whitham, 1967, 1974; Broer, 1975; Kaup, 1975; Kupershmidt, 1985; Ablowitz and Clarkson, 1991; Cox et al., 1991; Wang et al., 1996; Wang, 1995; Yan and Zhang, 1999), homotopy perturbation method (HPM) (Ganji and Rajabi, 2006; He, 1999b, 2000, 2005a, 2005b, 2005c), homotopy analysis (HAM) (He, 2006a), Adomian decomposition method (ADM) (Adomian, 1994; Wazwaz, 2002), variational iteration method (VIM) (Abdou and Soliman, 2005; He, 1998a, 1998b, 1999a, 2006b; He and Wu, 2006) and optimal homotopy asymptotic method (Haq and Ishaq, 2012; Ishaq and Haq, 2014).

Perturbation methods (Awrejcewicz et al., 1985; Acton and Squire, 1985) have been used for the solution of nonlinear problems in science and engineering. But, the following limitations restrict the applicability of these methods:

Most of these methods require the existence of a small parameter in the equation. But majority of nonlinear equations, especially those having strong nonlinearity, have no such parameter. An unsuitable choice of such parameter would lead to very bad results. It is not that easy to chose this parameter. The solutions obtained through perturbation methods can be valid only when a small value of the parameter is used. Hence, it is necessary to check validity of the approximations through experimental and/or numerical processes (Lee et al., 1997).

In HPM, HAM and OHAM, the concept of homotopy from topology and conventional perturbation technique were merged to propose a general analytic procedure for the solution of nonlinear problems. Thus, these methods are independent of the existence of a small parameter in the problem at hand and thereby overcome the limitations of conventional perturbation technique. OHAM, however, is the most generalized form of the remaining two as it employs a more general auxiliary function $H(p)$ in place of HPM's $-q$ and HAM's $\hbar q$.

OHAM was developed by Marinca et al. (2008, 2009) and Marinca and Herisanu (2008). The concept of auxiliary function in OHAM was taken from homotopy continuation method (HCM) and homotopy analysis method (HAM). For a detailed discussion the reader is referred to Marinca and Herisanu (2011). Instead of infinite series as is required in HAM, one needs only a few terms in OHAM, mostly two or three terms. This is because of the use of convergence control constants used in the auxiliary function. The OHAM is equally effective for ordinary and partial differential equations. Herisanu et al. used OAHM to determine analytical treatment of nonlinear vibration of an electrical machine and oscillators with discontinuities and fractional-power restoring force (Herisanu et al., 2008; Herisanu and Marinca, 2010a, 2010b). Solution of stagnation point flow with heat transfer analysis, and Couette and Poiseuille flows for fourth grade fluid were obtained using OHAM by Shah et al. (2010a, 2010b). Joneidi et al. managed to apply the method to micropolar flow in a

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porous channel with high mass transfer (Joneidi et al., 2009), Javed Ali et al. used the method to multipoint boundary value problems (Ali et al., 2010) and solutions of Jeffery–Hamel flow problem were obtained by Esmaeilpour and Ganji (2010). OHAM solutions were obtained for Couette and Poiseuille flows of a third grade fluid in Islam et al. (2010) and mixed convection flow past a vertical plate was solved with OHAM in Babaelahi et al. (2010). The technique was also used for the solution of special fourth- and sixth-order problems (Idrees et al., 2010b, 2010c). Linear and nonlinear Klein–Gordon (Iqbal et al., 2010) and squeezing flow problem (Idrees et al., 2010a) were studied using this method. This shows that OHAM has its validity and potential for the solution of linear nonlinear problems in science and engineering applications.

In this paper, we apply OHAM to solve the coupled Whitham–Broer–Kaup (WBK) equations

$$\frac{\partial \mathrm{U}^\delta}{\partial t}=(\delta-2)\left[\mathrm{U}^\delta\frac{\partial \mathrm{U}^\delta}{\partial x}+\frac{\partial \mathrm{U}^{\delta+1}}{\partial x}\right]+(-1)^{2-\delta}\gamma\frac{\partial^2 \mathrm{U}^\delta}{\partial x^2}+(1-\delta)\left[\frac{\partial(\mathrm{U}^{\delta-1}\mathrm{U}^\delta)}{\partial x}+\sigma\frac{\partial^3 \mathrm{U}^{\delta-1}}{\partial x^3}\right],\quad \delta=1,2 \tag{1}$$

where $\mathrm{U}^\delta=\mathrm{U}^\delta(x,t)$ represents horizontal velocity field and height deviating from the equilibrium position respectively. The γ,σ are constants representing different diffusion powers. Eqs. (1) are the governing equations of dispersive long wave in shallow water. If $\gamma=1$, $\sigma=0$, Eqs. (1) reduce to modified Boussinesq equation, and if $\gamma=0$, $\sigma=1$, the system represents classical long wave equation representing shallow water wave with dispersion.

The rest of the paper is organized as follows: In Section 2, the proposed method is described. OHAM solutions of the problem are given in Section 3, whereas Section 4 is devoted to the conclusion.

2. Description of the technique for coupled problems

In this section, we give an outline of the proposed method for a coupled system. For this, let us consider the following coupled equations:

$$\mathcal{L}(\mathrm{U}^\delta(x,t))+f^\delta(x)+\mathcal{N}(\mathrm{U}^\delta(x,t))=0,\quad x\in\Omega \tag{2}$$

$$\mathcal{B}\left(\mathrm{U}^\delta,\frac{\partial \mathrm{U}^\delta}{\partial x}\right)=0,\quad x\in\partial\Omega,\quad \delta=1,2 \tag{3}$$

where $\mathcal{L}$ and $\mathcal{N}$ are linear and nonlinear operators respectively, f^δ are known functions, U^δ are unknown functions and $\mathcal{B}$ is a boundary operator.

Thus OHAM is given by Marinca et al. (2008, 2009), Haq and Ishaq (2012), Marinca and Herisanu (2008)

$$(1-p)[\mathcal{L}(\phi^\delta(\eta,p))+f^\delta(\eta)]=\mathrm{H}_\delta(p)[\mathcal{L}(\phi^\delta(\eta,p))+f^\delta(\eta)+\mathcal{N}(\phi^\delta(\eta,p))] \tag{4}$$

$$\mathcal{B}\left(\phi^\delta(\eta,p),\frac{\partial\phi^\delta(\eta,p)}{\partial\eta}\right)=0,\quad \delta=1,2 \tag{5}$$

where $p\in[0,1]$ is an embedding operator. The functions ϕ^δ are unknown and H_δ are auxiliary functions such that

$$\mathrm{H}_\delta(p)=\begin{cases}\sum_{j=1}^{\infty}\mathrm{C}_j^\delta p^j & \text{if } p\neq 0\\ 0 & \text{if } p=0.\end{cases} \tag{6}$$

For our computational work we use

$$\mathrm{H}_\delta(p)=\begin{cases}\sum_{j=1}^{m}\mathrm{C}_j^\delta p^j & \text{if } p\neq 0\\ 0 & \text{if } p=0,\end{cases} \tag{7}$$

where C_j^δ's $(j=1,2,\ldots,m)$ are constants to be determined.

According to the new developments of OHAM, in order to increase the accuracy of the results (if necessary) and also for a fast convergence, the last term of the sum within Eqs. (7) could involve a physical parameter (such as the time) (Herisanu and Marinca, 2010a, 2010b; Marinca and Herisanu, 2010). Thus

$$\mathrm{H}_\delta(p)=\begin{cases}\sum_{j=1}^{m-1}\mathrm{C}_j^\delta p^j+p^m\vartheta^\delta(t) & \text{if } p\neq 0\\ 0 & \text{if } p=0.\end{cases} \tag{8}$$

In our case

$$\vartheta^\delta(t)=\mathrm{C}_m^\delta+\mathrm{C}_{m+1}^\delta t. \tag{9}$$

It may be noted that in the text below the notation OHAM_1 will be using when getting the solution with $\mathrm{H}_\delta(p)$ given in Eq. (7). Similarly OHAM_2 means that we are using Eq. (8). By definition of homotopy

$$\phi^\delta(\eta,0)=\mathrm{U}_0^\delta(\eta),\quad \phi^\delta(\eta,1)=\mathrm{U}^\delta(\eta) \tag{10}$$

where U_0^δ are obtained from Eqs. (4) and (5) when $p=0$, i.e., for $p=0$

$$\mathcal{L}(\mathrm{U}_0^\delta(\eta))+f^\delta(\eta)=0,\quad B\left(\mathrm{U}_0^\delta,\frac{\partial \mathrm{U}_0^\delta}{\partial\eta}\right)=0. \tag{11}$$

Now by Taylor's series, we have

$$\phi^\delta(\eta,p,\mathrm{C}_j^\delta)=\mathrm{U}_0^\delta(\eta)+\sum_{k\geq 1}\mathrm{U}_k^\delta(\eta,\mathrm{C}_j)p^k,\quad j=1,2\ldots,m \tag{12}$$

where

$$\mathrm{U}_k^\delta(\eta,\mathrm{C}_j^\delta)=\frac{1}{k!}\frac{\partial^k\phi^\delta(\eta,p,\mathrm{C}_j^\delta)}{\partial p^k}\bigg|_{p=0} \tag{13}$$

Using Eqs. (7) and (12) in Eqs. (4), (5) and equating like powers of p we have

$$\mathcal{L}(\mathrm{U}_1^\delta(\eta))=\mathrm{C}_1^\delta\mathcal{N}_0(\mathrm{U}_0^\delta(\eta)),\quad B\left(\mathrm{U}_1^\delta,\frac{\partial \mathrm{U}_1^\delta}{\partial\eta}\right)=0 \tag{14}$$

$$\mathcal{L}(\mathrm{U}_k^\delta(\eta)-\mathrm{U}_{k-1}^\delta(\eta))=\mathrm{C}_j^\delta\mathcal{N}_0(\mathrm{U}_0^\delta(\eta))+\sum_{j=1}^{k-1}\mathrm{C}_j^\delta[\mathcal{L}(\mathrm{U}_{k-j}^\delta(\eta))+\mathcal{N}_{k-j}(\mathrm{U}_0^\delta(\eta),\mathrm{U}_1^\delta(\eta),\ldots,\mathrm{U}_{k-j}^\delta(\eta))],$$

$$\mathcal{B}\left(\mathrm{U}_k^\delta,\frac{\partial \mathrm{U}_k^\delta}{\partial\eta}\right)=0,\quad k=2,3,\ldots,m, \tag{15}$$

where $\mathcal{N}_m(\mathrm{U}_0^\delta(\eta),\mathrm{U}_1^\delta(\eta),\ldots,\mathrm{U}_m^\delta(\eta))$ is the coefficient of p^m, in the expansion of $N(\phi^\delta(\eta,p,\mathrm{C}_j^\delta))$ with respect to the embedding parameter p (Marinca and Herisanu, 2014):

$$\mathcal{N}(\phi^\delta(\eta,p,\mathrm{C}_j^\delta))=\mathcal{N}_0(\mathrm{U}_0^\delta(\eta))+\sum_{m\geq 1}\mathcal{N}_m(\mathrm{U}_0^\delta(\eta),\mathrm{U}_1^\delta(\eta),\ldots,\mathrm{U}_m^\delta(\eta)),\quad j=1,2,\ldots\,. \tag{16}$$

It should be noted that U_k^δ, for $k\geq 1$, are governed by Eqs. (11)–(15) which also involve their linear boundary conditions derived from the original problem and can be solved easily.

The convergence of the series (12) depend upon the auxiliary constants $\mathrm{C}_j^\delta, j\in\mathrm{N}$. Putting $p=1$ in Eqs. (12), we obtain

$$\mathrm{U}^\delta(\eta,\mathrm{C}_j^\delta)=\mathrm{U}_0^\delta(\eta)+\sum_{k\geq 1}\mathrm{U}_k^\delta(\eta,\mathrm{C}_j^\delta),\quad j=1,2,\ldots \tag{17}$$

In actual calculation

$$(\mathrm{U}^\delta)^{(m)}(\eta,\mathrm{C}_j^\delta)=\mathrm{U}_0^\delta(\eta)+\sum_{k=1}^{m}\mathrm{U}_k^\delta(\eta,\mathrm{C}_j^\delta),\quad j=1,2,\ldots \tag{18}$$

Substituting Eqs. (18) into Eq. (2), we get the following residual:

$$\mathrm{R}^\delta(\eta,\mathrm{C}_j^\delta)=\mathcal{L}((\mathrm{U}^\delta)^{(m)}(\eta,\mathrm{C}_j^\delta))+f^\delta(\eta)+\mathcal{N}((\mathrm{U}^\delta)^{(m)}(\eta,\mathrm{C}_j^\delta)),\quad j=1,2 \tag{19}$$

When $\mathrm{R}^\delta(\eta,\mathrm{C}_j^\delta)=0$, then $((\mathrm{U}^\delta)^{(m)}(\eta,\mathrm{C}_j^\delta))$ correspond to the exact solution.

However, when $\mathrm{R}^\delta(\eta,\mathrm{C}_j^\delta)\neq 0$, then taking $k_i\in\Omega, i=1,2,\ldots,m$ and substituting $\eta=k_i$ in Eq. (19) equating to zero, we having the