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Short Communication

First integral method to look for exact solutions of a variety of Boussinesq-like equations

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ABSTRACT

In this paper, we study a variety of Boussinesq-like equations. The first integral method is applied to obtain exact 1-soliton solutions for each equation. Exact 1-soliton solutions of these equations are found.

1. Introduction

Finding the exact solutions of nonlinear evolution equations (NLEEs) plays an important role in the study of many physical phenomena in various fields such as fluid mechanics, solid-state physics, plasma physics, chemical physics, optical fibre and geochemistry. Thus, it is important to investigate the exact explicit solutions of NLEEs (Ahmed and Biswas, 2013; Biswas, 2013; Biswas et al., 2009, 2011; Ebadi et al., 2012, 2013; Krishnan et al., 2012; Razborova et al., 2013, 2014; Triki et al., 2012).

The first integral method was first proposed in Feng (2002) for obtaining the exact 1-soliton solutions of the Burger-KdV equation which is based on the ring theory of commutative algebra. The basic idea of this method is to construct a first integral with polynomial coefficients of an explicit form to an equivalent autonomous planar system by using the division theorem. Recently, this useful method is widely used in many papers such as in (Aslan, 2011, 2012; Bekir and Unsal, 2012; Jafari et al., 2012; Lu, 2012; Mirzazadeh and Eslami, 2012; Eslami and Mirzazadeh, 2013; Tascan et al., 2009) and the references therein.

In this paper, we will consider the three different forms of Boussinesq equations (Wazwaz, 2012) as follows:

$$u_{tt}-(6u^2u_x+u_{xxx})_x=0, \tag{1}$$

$$u_{tt}-u_{xx}-(6u^2u_x+u_{xtt})_x=0, \tag{2}$$

$$u_{tt}-u_{xt}-(6u^2u_x+u_{xxt})_x=0. \tag{3}$$

Eqs. (1)–(3) model the study of shallow water waves in lakes and ocean beaches. More details are presented in Wazwaz (2012).

Also, we will consider the following (1 + 1)-dimensional nonlinear Boussinesq equations:

$$u_t+\alpha_1 v_x+\alpha_2 uu_x=0,$$
$$v_t+\alpha_3(vu)_x+\alpha_4 u_{xxx}=0. \tag{4}$$

This coupled Boussinesq equation arises in shallow water waves for two layered fluid flow. This situation occurs when there is an accidental oil spill from a ship which results in a layer of oil floating above the layer of water. More details are presented in Jawad et al. (2013).

In this paper, we would like to obtain the exact 1-soliton solutions of Eqs. (1)–(4) by using the first integral method.

2. First integral method

Tascan et al. (2009) summarized the main steps for using the first integral method as follows:

Step 1. Suppose that a nonlinear PDE

$$P(u,u_t,u_x,u_{tt},u_{xt},u_{xx},\ldots)=0 \tag{5}$$

can be converted to an ODE

$$Q(U,-\omega U',kU',\omega^2U'',-k\omega U'',k^2U'',\ldots)=0, \tag{6}$$

using a traveling wave variable $u(x,t)=U(z)$, $z=kx-\omega t$, where the prime denotes the derivation with respect to z. If all terms contain derivatives, then Eq. (6) is integrated where integration constants are considered zeros.

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Step 2. Suppose that the solution of ODE (6) can be written as follows:

$$u(x,t)=U(z)=f(z). \tag{7}$$

Step 3. We introduce a new independent variable

$$X(z)=f(z), \quad Y(z)=f_z(z), \tag{8}$$

which leads a system of

$$X_z(z)=Y(z),$$

$$Y_z(z)=F(X(z),Y(z)). \tag{9}$$

Step 4. According to the qualitative theory of ordinary differential equations (Ding and Li, 1996), if we can find the integrals to (9) under the same conditions, then the general solutions to (9) can be solved directly. However, in general, it is really difficult for us to realize this even for one first integral, because for a given plane autonomous system, there is neither a systematic theory that can tell us how to find its first integrals, nor a logical way for telling us what these first integrals are. We shall apply the Division Theorem to obtain one first integral to (9) which reduces (6) to a first-order integrable ordinary differential equation. An exact solution to (5) is then obtained by solving this equation. Now, let us recall the Division Theorem:

Division Theorem. Suppose that $P(w,\nu)$ and $Q(w,\nu)$ are polynomials in $C[w,\nu]$; and $P(w,\nu)$ is irreducible in $C[w,\nu]$. If $Q(w,\nu)$ vanishes at all zero points of $P(w,\nu)$, then there exists a polynomial $G(w,\nu)$ in $C[w,\nu]$ such that

$$Q(w,\nu)=P(w,\nu)G(w,\nu).$$

3. The first Boussinesq-like equation

We first study the first Boussinesq-like equation given by

$$u_{tt}-(6u^2u_x+u_{xxx})_x=0. \tag{10}$$

By making the transformation

$$u(x,t)=U(z), \quad z=kx-\omega t \tag{11}$$

Eq. (10) becomes

$$\omega^2U''(z)-k(6kU^2(z)U'(z)+k^3U'''(z))'=0. \tag{12}$$

Integrating Eq. (12) twice and setting the integration constants to zero yield

$$\omega^2U(z)-2k^2U^3(z)-k^4U''(z)=0. \tag{13}$$

Introducing new variables $X=U(z)$ and $Y=U'(z)$ converts Eq. (13) into a system of ODEs

$$X'(z)=Y(z),$$
$$Y'(z)=\frac{\omega^2}{k^4}X(z)-\frac{2}{k^2}X^3(z). \tag{14}$$

Now, the Division Theorem is employed to seek the first integral to Eq. (14). Suppose that $X(z)$ and $Y(z)$ are nontrivial solutions of Eqs. (14) and

$$Q(X,Y)=\sum_{i=0}^{m}a_i(X)Y^i=0$$

is an irreducible polynomial in the complex domain $C[X,Y]$ such that

$$Q(X(z),Y(z))=\sum_{i=0}^{m}a_i(X(z))Y^i(z)=0, \tag{15}$$

where $a_i(X)$ $(i=0,1,\ldots,m)$ are polynomials of X and $a_m(X)\neq 0$. Due to the Division Theorem, there exists a polynomial $g(X)+h(X)Y$ in the complex domain $C[X,Y]$ such that

$$\frac{dQ}{dz}=\frac{dQ}{dX}\frac{dX}{dz}+\frac{dQ}{dY}\frac{dY}{dz}$$
$$=(g(X)+h(X)Y)\sum_{i=0}^{m}a_i(X)Y^i. \tag{16}$$

Suppose that $m=1$, by equating the coefficients of Y^i $(i=2,1,0)$ on both sides of Eq. (16), we have

$$a_1'(X)=h(X)a_1(X), \tag{17}$$

$$a_0'(X)=g(X)a_1(X)+h(X)a_0(X), \tag{18}$$

$$a_1(X)\left[\frac{\omega^2}{k^4}X-\frac{2}{k^2}X^3\right]=g(X)a_0(X). \tag{19}$$

Since $a_i(X)$ $(i=0,1)$ are polynomials, from (17) we deduce that $a_1(X)$ is constant and $h(X)=0$. For simplicity, take $a_1(X)=1$. Balancing the degrees of $g(X)$ and $a_0(X)$, we conclude that $\deg(g(X))=1$ only. Suppose that $g(X)=A_1X+B_0$, then we find that

$$a_0(X)=A_0+B_0X+\tfrac{1}{2}A_1X^2, \tag{20}$$

where A_0 is an arbitrary integration constant.

Substituting $a_0(X)$ and $g(X)$ into (19) and setting all the coefficients of powers X to be zero, we obtain a system of nonlinear algebraic equations and by solving it, we obtain

$$B_0=0, \quad A_0=-\frac{i\omega^2}{2k^3}, \quad A_1=\frac{2i}{k}, \tag{21}$$

$$B_0=0, \quad A_0=\frac{i\omega^2}{2k^3}, \quad A_1=-\frac{2i}{k}, \tag{22}$$

where k and ω are arbitrary constants.

Using the conditions (21) and (22) in Eq. (15), we obtain

$$Y(z)\mp\left(\frac{i\omega^2}{2k^3}-\frac{i}{k}X^2(z)\right)=0. \tag{23}$$

Combining these equations with (14), we obtain the exact solutions to (13) and then the exact solutions to Eq. (10) which can be written as

$$u_{1,2}(x,t)=\pm\frac{\omega}{k\sqrt{2}}\tanh\left(\frac{\omega}{i\sqrt{2}k^2}(kx-\omega t+z_0)\right), \tag{24}$$

and

$$u_{3,4}(x,t)=\pm\frac{\omega}{k\sqrt{2}}\coth\left(\frac{\omega}{i\sqrt{2}k^2}(kx-\omega t+z_0)\right), \tag{25}$$

where z_0 is an arbitrary constant.

4. The second Boussinesq-like equation

We next study the second Boussinesq-like equation given by

$$u_{tt}-u_{xx}-(6u^2u_x+u_{xtt})_x=0. \tag{26}$$

The wave transformation $u(x,t)=U(z)$, $z=kx-\omega t$, reduces Eq. (26) to the following ODE:

$$(\omega^2-k^2)U''(z)-k(6kU^2(z)U'(z)+k\omega^2U'''(z))'=0. \tag{27}$$

Integrating Eq. (27) twice and setting the integration constants to zero, we get

$$(\omega^2-k^2)U(z)-2k^2U^3(z)-k^2\omega^2U''(z)=0. \tag{28}$$