Annals of Nuclear Energy 89 (2016) 84–89

Contents lists available at ScienceDirect

Annals of Nuclear Energy

journal homepage: www.elsevier.com/locate/anucene

A numerical solution to the nonlinear point kinetics equations using Magnus expansion

Yun Cai [a,b,*], Xingjie Peng [a,b], Qing Li [b], Kan Wang [a]

[a] Department of Engineering Physics, Tsinghua University, Beijing 100084, China
[b] Science and Technology on Reactor System Design Technology Laboratory, Nuclear Power Institute of China, Chengdu 610041, China

ARTICLE INFO

Article history:
Received 26 January 2015
Received in revised form 9 November 2015
Accepted 12 November 2015
Available online 8 December 2015

Keywords:
Point kinetics equations
Magnus expansion
Newtonian temperature feedback

ABSTRACT

The solution of point kinetics equations can predict the neutron density variation during the operation of a nuclear reactor, which is very important in reactor safety and analysis. The point kinetics equations are coupled nonlinear ordinary differential equations, which are stiff and difficult to solve. In this paper, the Magnus expansion is proposed to solve the neutron point kinetics equations. It is convenient to construct the exponential form of the solution with the Magnus expansion. The present method is tested using a variety of cases with step, ramp, sinusoidal and temperature feedback reactivity insertions. The results indicate that the method is accurate for thermal and fast reactors.

1. Introduction

In nuclear reactor dynamics, the point kinetics equations describe the variation of the neutron density and delayed neutron precursor concentrations as the reactivity changes. The equations become complex and nonlinear when the reactivity depends on the neutron density and it is the common situation when temperature feedback arises. As the prompt and delayed neutron lifetimes are different in the magnitude orders, the system of the kinetics equations is stiff. Small time steps are required in many numerical methods to alleviate the stiffness.

Many methods for solving point kinetics have been proposed over the past 50 years. Chao and Attard (1985) proposed the stiffness confinement method which confines the stiffness to the equation for the prompt neutron. Aboanber and Nahla (2002b) described Padé approximations for solving the point reactor kinetics equations with multi group of delayed neutrons with constant, ramp and temperature feedback reactivity. They (2002a, 2003) also obtained the solution of a point kinetic equation using exponential mode analysis and generalization of analytical inversion method. Li et al. (2009) presented the better basis function method (BBF) for solving the point kinetics equations and investigated the power transients caused by various types of reactivity insertion for thermal reactors with multi group of delayed neutrons.

Kinard and Allen (2004) proposed the numerical solution of the point kinetics equations based on piecewise constant approximation (PCA). They used average value to approximate the varied reactivity in one time step, and then solved the simplified equations exactly, and they proved that the method converges with order 2. Picca et al. (2013) solved the point kinetics equations with a new technique which is based on PCA. They tackled the error term into the source term, and iteratively corrected it. This method is called enhanced piecewise constant approximation (EPCA), and showed extreme accuracy.

McMohan and Pierson (2010) used Taylor series method (TSM), and Nahla (2011) applied TSM to solve the point kinetics equations with the Newtonian temperature feedback reactivity. Kim et al. (2014) described a numerical solution to the point kinetics equations using Taylor–Lie series combined with a scaling and squaring technique. Ganapol (2013) combined implicit backward Euler finite difference (BEFD) with Richardson extrapolation technique which can obtain a highly accurate solution. Patra and Ray (2014) introduced Haar wavelet operational method (HWOM) for solving the point kinetics equations. They expanded the solution in Haar wavelet series in one time step and integrated Haar wavelet.

The Magnus expansion has been successfully applied in many areas of physics and chemistry, from classical Hamilton mechanics to atomic and molecular physics, nuclear magnetic resonance, quantum electrodynamics, and time-independent Schrödinger equation (Blanes et al., 2009). On the mathematical side, it can be used as an efficient numerical integrator.

* Corresponding author at: Department of Engineering Physics, Tsinghua University, Beijing 100084, China.
E-mail address: caiyun12@mails.tsinghua.edu.cn (Y. Cai).

http://dx.doi.org/10.1016/j.anucene.2015.11.021
0306-4549/

Magnus expansion is a widespread tool to construct approximate exponential representations of the solution of the non-autonomous linear differential equations (Casas and Iserles, 2006). The exponent of the solution in Magnus expansion is expanded as an infinite series. It is well known that PCA is generally inexact unless the time-dependent reactivity is step function. In fact, PCA can be viewed as the special Magnus expansion when the exponent is approximated by the first term. And it is convenient to construct the high order method using Magnus expansion.

In this present work, Magnus expansion is presented for solving the point kinetics equation without/with Newtonian temperature feedback effects. The numerical results of this method are compared with other methods for numerous cases of different reactivity insertion.

2. The point kinetics equations

The point reactor kinetics equations with multi-group delayed neutrons and Newtonian temperature feedback reactivity are the stiff nonlinear ordinary differential equations as follows:

$$\frac{dn}{dt}=\frac{\rho-\beta}{\Lambda}n(t)+\sum_{i=1}^{m}\lambda_i c_i \tag{1}$$

$$\frac{dc_i}{dt}=\frac{\beta_i}{\Lambda}n(t)-\lambda_i c_i,\quad i=1,\ldots,m \tag{2}$$

$$\rho(t)=\gamma(t)-\alpha(T(t)-T_0) \tag{3}$$

$$\frac{dT(t)}{dt}=K_c n(t) \tag{4}$$

where n is the neutron density, c_i is the ith precursor density, ρ is the time-dependent reactivity, β_i is the ith delayed fraction and β is the total delayed fraction, Λ is the neutron generation time, and λ_i is the ith group decay constant, m is the total number of delayed neutron groups. T is the temperature of the reactor, T_0 is the initial value of the temperature, $\gamma(t)$ is the prescribed reactivity, α is the temperature coefficient of reactivity, and K_c is the reciprocal of the thermal capacity of reactor.

The system can be simplified if the Eq. (3) is differentiated and substituted into Eq. (4)

$$\frac{d\rho(t)}{dt}=\frac{d\gamma(t)}{dt}-\alpha K_c n(t)=\frac{d\gamma(t)}{dt}-Bn(t) \tag{5}$$

The final nonlinear reactor kinetic model is described by the Eqs. (1)–(3), and (5). Here, the equation of the temperature is not solved explicitly because there is a simple algebraic relationship between the temperature and the reactivity function. The nonlinear system in the compact matrix form is written as follows:

$$\frac{d}{dt}y(t)=A_{NL}(t,y)y(t)+q(t),\ A_{NL}(t,y)=\begin{pmatrix}\frac{\rho-\beta}{\Lambda} & \lambda_1 & \cdots & \lambda_N & 0\\ \frac{\beta_1}{\Lambda} & -\lambda_1 & & & \\ \vdots & & \ddots & & \\ \frac{\beta_m}{\Lambda} & & & -\lambda_m & \\ -B & & & & 0\end{pmatrix},$$

$$y=\begin{bmatrix}n\\ c_1\\ \vdots\\ c_m\\ \rho\end{bmatrix},\quad q(t)=\begin{bmatrix}0\\ 0\\ \vdots\\ 0\\ \frac{d}{dt}\gamma(t)\end{bmatrix} \tag{6}$$

However, it can be written in another form which is more convenient to deal with. By increasing one dimension, the system can be described equivalently as following:

$$\frac{d}{dt}\begin{pmatrix}y(t)\\ 1\end{pmatrix}=\begin{pmatrix}A_{NL}(t,y) & q(t)\\ 0 & 0\end{pmatrix}\begin{pmatrix}y(t)\\ 1\end{pmatrix} \tag{7}$$

The linear reactor kinetic model can be viewed as the special case of the nonlinear system. However, it will increase unnecessary computation. Here, the linear reactor kinetic model is viewed as the different model, and described similarly as Eq. (7)

$$\frac{d}{dt}z(t)=A_L(t)z(t),\ A_L(t)=\begin{pmatrix}\frac{\rho(t)-\beta}{\Lambda} & \lambda_1 & \cdots & \lambda_N\\ \frac{\beta_1}{\Lambda} & -\lambda_1 & & \\ \vdots & & \ddots & \\ \frac{\beta_N}{\Lambda} & & & -\lambda_N\end{pmatrix},\quad z=\begin{bmatrix}n\\ c_1\\ \vdots\\ c_m\end{bmatrix} \tag{8}$$

where the reactivity function $\rho(t)$ is the known function and equals $\gamma(t)$.

The initial condition of the linear/nonlinear reactor kinetic model is that $\rho(0)=\rho_0, n(0)=n_0, c_i(0)=n_0\beta_i/(\lambda_i\Lambda)$.

3. The Magnus expansion

Here, the following model is considered which contains the linear and nonlinear reactor kinetic model:

$$\frac{d}{dt}Y(t)=A(t,Y)Y(t) \tag{9}$$

According to the literature (Casas and Iserles, 2006), if the solution above is represented in the form of matrix exponential

$$Y(t)=e^{\Omega(t)}Y_0 \tag{10}$$

then Ω satisfies the differential equation by the means of the Lie algebra:

$$\frac{d}{dt}\Omega(t)=d\exp_\Omega^{-1}(A(t,e^{\Omega(t)}Y_0)),\Omega(0)=0. \tag{11}$$

Here

$$d\exp_\Omega^{-1}(C)=\sum_{k=0}^{\infty}\frac{B_k}{k!}ad_\Omega^k C, \tag{12}$$

where the start time is considered to be 0, B_k is the kth Bernoulli number and the definition of ad^m can be recursively written:

$$ad_\Omega^0 C=C,\quad ad_\Omega^{m+1}C=[\Omega,ad_\Omega^m C] \tag{13}$$

where $[X, Y] = XY - YX$ is the commutator of X and Y. Eq. (11) is solved by Picard's iteration and Ω is approximated

$$\Omega^{[m+1]}(t)=\int_0^t\sum_{k=0}^{\infty}\frac{B_k}{k!}ad_{\Omega^{[m]}(s)}^k A(s,e^{\Omega^{[m]}(s)}Y_0)ds,\Omega^{[0]}(t)=0 \tag{14}$$

However, the summation needs to be truncated, and the practical implement is the formulas:

$$\begin{aligned}\Omega^{[1]}(t)&=\int_0^t A(s,Y_0)ds\\ \Omega^{[m]}(t)&=\sum_{k=0}^{m-2}\frac{B_k}{k!}\int_0^t ad_{\Omega^{[m-1]}(s)}^k A\left(s,e^{\Omega^{[m-1]}(s)}Y_0\right)ds,\ m\geqslant 2\end{aligned} \tag{15}$$

According to the literature (Casas and Iserles, 2006), the formulas above can achieve the order up to m

$$\Omega^{[m]}(t)-\Omega(t)=O(t^{m+1}) \tag{16}$$