Annals of Nuclear Energy 86 (2015) 35–44

Contents lists available at ScienceDirect

Annals of Nuclear Energy

journal homepage: www.elsevier.com/locate/anucene

Time-dependent 2-stream particle transport

Noel Corngold

T.J. Watson Laboratories of Applied Physics, California Institute of Technology, Pasadena, CA 91125, USA

ARTICLE INFO

Article history:
Received 27 August 2014
Received in revised form 18 March 2015
Accepted 22 March 2015

Keywords:
Two-stream transport
Stochastics

ABSTRACT

We consider time-dependent transport in the 2-stream or "rod" model via an attractive matrix formalism. After reviewing some classical problems in homogeneous media we discuss transport in materials whose density may vary. There we achieve a significant contraction of the underlying Telegrapher's equation. We conclude with a discussion of stochastics, treated by the "first-order smoothing approximation."

Mike's heroic 40 years as editor of Annals of Nuclear Energy has earned him a place in the Pantheon of Great Editors, along with Sam Goudsmit of Physical Review (a mere 25 years) and S. Chandrasekhar of The Astrophysical Journal (20 years). In an age of scholarly excess he has kept high the standards of our field, walled it from the slack and sloppy.

How he has managed this task, while producing several hundred scientific papers of distinction, along with books and lecture-notes remains, after many years, a mystery. Either, hidden in his bloodline is a Hungarian (therefore Martian, and extraplanetary-) strain which he shares with his talented cousins (Wigner, Teller, Szilard, von Neumann..) or as I theorized some years ago, Mike has access to a special source of energy, perhaps nuclear, not mundane biochemical, forbidden us mortals.

Cheers, Mike, and thank you for four decades of service and a half-century of friendship.

1. Introduction

There has been continued and significant concern in the transport community about transport of neutrons and radiant energy in unusual media, or in regular arrangements subject to randomness. In nuclear engineering one might have a bed of randomly arranged fuel particles as a host (Vasques and Larsen, 2014), or a boiling moderator; in bio-physics one encounters near-infra-red radiation diffusing through the complexities of animal tissue. And there are the active fields of "ocean-optics" and radiation transport through atmospheres.

Introducing stochasticity into the mesoscopic equations of transport theory is a second application of that concept, for the transport equations themselves result from a stochastic treatment of the exact equations of motion. This second averaging deals with fluctuations on a larger scale and increases the calculational challenge. In the last several decades heroic and sophisticated response to the challenge has been made by colleagues (Akcasu, Larsen, Prinja, Sanchez, Williams…) stimulated, perhaps, by Gerry Pomraning's 1991 monograph (Pomraning, 1991). Mike Williams' many contributions have been rich and broad. For examples, see (Williams, 2000a,b, 2006).

Our concern here is with the subject of "tissue optics." (Tuchin, 2000) Its connection with medicine at both the research and clinical levels has stimulated a great amount of activity (Vo-Dinh, 2003). To a transport theorist, the tools used there in analysis and design appear somewhat naïve. The purpose of our essay is to comment on the innocent use of diffusion theory that prevails and to recommend a model, hardly new, that may prove superior in many situations – particularly when one probes dynamically.

In bio-photonics, coherent radiation in the near-infra-red range (0.6–1.0 microns) is delivered by a laser onto tissue. In that range, absorption is low and one can, by reflection or transmission, probe an occult structure. Or, one can generate, perhaps by fluorescence, interesting secondary radiation. After a few collisions, coherence is lost (Ishimaru, 1978; Corngold, 2012) and the radiation intensity is governed by a transport equation that may be quite complicated. The material is heterogeneous on many scales! Further modeling is needed and issues arise. All agree that two features of the scattering process must be preserved in simplified models – that the scattering is elastic and that it is strongly forward-peaked. Most workers prefer a transport model that is a diffusion equation, one that is characterized by three parameters μ_a, μ_s, g, describing absorption and scattering.

But which diffusion equation? There is no debate about the absorption term – the issue is the quantity appearing as a

http://dx.doi.org/10.1016/j.anucene.2015.03.037
0306-4549/

"diffusion co-efficient." If one makes the highly questionable assumption that the angular flux is close to isotropic, the co-efficient is, in a weak absorber, simply a mean-free path divided by the quantity (1 – g), where g is the mean cosine of the scattering angle. Since g is quite close to unity for most tissues, such a result should make one uneasy and question the analysis. Yet the biophysics literature is rich with tables of g to be used for various tissues, from brain to spleen to teeth (Jacques, 2013), and calculations proceed happily.

Some years ago, (Aronson and Corngold, 1999) the tissue-optics community was enlightened as to "what every reactor-theorist knows," that the proper generation of a diffusion (Helmholtz) equation stems from examination of the "asymptotic regime," where transport is described by the dominant discrete eigenvalue of the appropriate transport equation. One extracts from that eigenvalue an (effective) "diffusion coefficient." The simple $\ell/(1-g)$ description has limited justification, particularly when absorption is present. Our comments were endorsed, subsequently, by experiment (Ripoll et al., 2005) and by computation (Hoenders and Graaff, 2005.) In this picture, careful analysis of benchmark experiments would require detailed knowledge of the eigen-value spectrum of the mono-energetic transport equation. Here, our knowledge is incomplete, alas.

The discussion which follows concerns another simple model, one that is, arguably, richer than the naïve g-model. It is the ancient, two-stream model which has an enormous literature attached to it. (One paper (Meador and Weaver, 1980), addressed to the atmospheric science community, sports more than 250 citations!) While the model is limited to slab geometry, and so to simple benchmark experiments and calculations, it deserves status in the toolbox of models. It is "causal," and can treat time-dependent phenomena on a scale which, though short, is absent from the time-dependent diffusion equation. It is equivalent to the transport equation in the "rod-model" limit. The equation for the total density is a rather general Telegrapher's equation. (For a delightful review of that popular equation, see Weiss (2002).)

In this note we treat a collection of problems, static and time-dependent, associated with finite media: reflection, transmission, inclusions. We give a general formulation of transport in non-uniform slabs and end with a short discussion of fluctuating media. The steady situation has been richly treated by others (Akcasu and Williams, 2004), (Akcasu, 2008); we emphasize time-dependent interrogation, which leads to wave motions and, ultimately, to the extraction of the response-function for the material. There is a discussion of the inhomogeneous host, in which we are able to reduce the burden of solving a challenging Telegrapher's equation significantly. Finally, the essay is in part pedagogical, indeed "promotional," in some ways a starting point. We endorse the treatment of finite media in a two-dimensional vector space, where the matrix formulation may be made compact and agreeable, and urge that this simple model be given a respectable status in the analyst's toolbox.

2. The transport equation

We consider the transport of particles whose energy is unaltered by scattering. The source and the medium are such that the particle density depends only upon a single spatial variable ("plane geometry.") Using conventional notation, we have

$$\left(\frac{1}{v}\frac{\partial}{\partial t'}+\mu\frac{\partial}{\partial x'}\right)\phi'(x',\mu,t')+n(x')[\sigma_a+\sigma_s]\phi'(x',\mu,t')-n(x')\sigma_s \int d\mu' p(\mu',\mu)\phi'(x',\mu',t')=Q'(x',\mu,t') \quad (1)$$

for transport in a scattering and absorbing medium of varying density. We write the density of the background material as $n(x')=n_0(1+\theta(x'))$, then convert to dimension-less variables through, $vn_0(\sigma_s+\sigma_a)t'=t$, $n_0(\sigma_s+\sigma_a)x'=x$, $\phi'(x',\mu,t')=vn_0(\sigma_s+\sigma_a)\phi(x',\mu,t')$. The result is the dimensionless equation

$$\left(\frac{\partial}{\partial t}+\mu\frac{\partial}{\partial x}\right)\phi(x,\mu,t)+(1+\theta(x))(1-cP)\phi=Q(x,\mu,t), \quad (2)$$

where $c=\frac{\sigma_s}{\sigma_s+\sigma_a}$, and the scattering operator has been denoted by P.

We convert to the 2-stream model, whereupon the flux becomes a 2-component vector $\phi=\begin{pmatrix}\phi_+\\ \phi_-\end{pmatrix}$ as does the source term. Then, $\phi_++\phi_-=\phi_0$ is the full flux, and $\phi_+-\phi_-=j$ the "current" (density.) The several operators become 2×2 matrices, denoted henceforth by various $\sigma's$ One may eliminate or combine components and go to a scalar description, which is a complicated version of the Telegrapher's equation. We choose an algebraic approach, with special matrices, projectors,

$$\sigma_*=\frac{1}{2}\begin{pmatrix}1&-1\\1&-1\end{pmatrix},\quad \sigma_*^T=\frac{1}{2}\begin{pmatrix}1&1\\-1&-1\end{pmatrix},$$

whose squares vanish, $\sigma_*^2=0$, and whose commutators are

$$[\sigma_*,\sigma_*^T]_+=1 \quad [\sigma_*,\sigma_*^T]=\sigma_x$$

(Recall the anti-commuting Pauli matrices – whose squares are the unit matrix,

$$\sigma_x=\begin{pmatrix}0&1\\1&0\end{pmatrix},\ \sigma_y=\begin{pmatrix}0&-i\\i&0\end{pmatrix},$$
$$\sigma_x=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad \sigma_x\sigma_y=-\sigma_y\sigma_x=i\sigma_z,\ \text{etc.})$$

The product, $\sigma_*\sigma_*^T=\frac{1}{2}\begin{pmatrix}1&1\\1&1\end{pmatrix}=\frac{1}{2}(1+\sigma_x)$ is a projector, $(\sigma_*\sigma_*^T)^2=\sigma_*\sigma_*^T$, projecting an arbitrary vector onto the unit vector. These matrices have the nice property that for arbitrary (vector) flux ϕ

$$\sigma_*\phi=(j/2)\begin{pmatrix}1\\1\end{pmatrix}\quad \sigma_*^T\phi=(\phi_0/2)\begin{pmatrix}1\\-1\end{pmatrix},$$

as well as

$$\sigma_*^T\sigma_*\phi=(j/2)\begin{pmatrix}1\\-1\end{pmatrix}\quad \sigma_*\sigma_*^T\phi=(\phi_0/2)\begin{pmatrix}1\\1\end{pmatrix},$$

$$\sigma_z=\sigma_*+\sigma_*^T,\quad \sigma_*\sigma_*^T\sigma_*=\sigma_*,$$

and

$$(A\sigma_*+B\sigma_*^T)^2=AB\underline{1}$$

The scattering is $P=\begin{pmatrix}p_1&p_2\\p_2&p_1\end{pmatrix}$ with $p_1+p_2=1$, p_1,p_2 describing forward and backward scattering, respectively, while g =(p_1-p_2) is the "averaged cosine of scattering." Other useful quantities are

$$c_1=cp_1, c_2=cp_2,\quad \alpha=1-cg,\quad \beta=1-c,$$
$$\frac{\alpha+\beta}{2}=1-c_1,\quad \frac{\alpha-\beta}{2}=c_2$$

While using α is convenient, it has the drawback that its value depends upon both absorption and anisotropy. As special cases, note that when the scattering is isotropic, $\alpha=1$ independent of absorption. In the absence of absorption, $\alpha=2p_2$. Otherwise $\alpha=(1-c)+2cp_2$.

In treating the two-stream transport equation it will be convenient to "divide-by-μ" that is, multiply the equation by σ_z. We then