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An analytical discrete-ordinates solution for an improved one-dimensional model of three-dimensional transport in ducts

R.D.M. Garcia

Instituto de Estudos Avançados, Trevo Cel. Av. José Alberto Albano do Amarante no 1, São José dos Campos, SP 12228-001, Brazil

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Dedicated to M.M.R. Williams for his many outstanding contributions to transport theory.

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ABSTRACT

An analytical discrete-ordinates solution is developed for the problem of particle transport in ducts, as described by a one-dimensional model constructed with two basis functions. Two types of particle incidence are considered: isotropic incidence and incidence described by the Dirac delta distribution. Accurate numerical results are tabulated for the reflection probabilities of semi-infinite ducts and the reflection and transmission probabilities of finite ducts. It is concluded that the developed solution is more efficient than commonly used numerical implementations of the discrete-ordinates method.

1. Introduction

Particle transport in ducts is inherently a three-dimensional (3-D) problem, and for this reason it has been usually approached with the Monte Carlo and view factor methods, which are computationally expensive. Looking for an economical alternative, Prinja and Pomraning (1984) introduced an approximate one-dimensional (1-D) model that is based on averaging the distance between particle–wall collisions over the duct surface and the azimuthal angle. The result of this averaging process was interpreted as a mean free path, the reciprocal of which was taken as an interaction cross section. This cross section was then used in a plane-geometry transport equation that has z, the position along the duct axis, and μ, the cosine of the polar angle, as independent variables. Such a transport equation has the unusual feature shared by a few other transport equations that have appeared in the literature (Williams, 1978; Shultis and Myneni, 1988; Myneni and Ganapol, 1991; Williams, 1992) of an angularly dependent cross section. Admittedly, the 1-D model of Prinja and Pomraning (1984) was derived in a heuristic way but soon it was put into firm mathematical grounds by Larsen (1984), who showed that the Prinja–Pomraning model corresponds to the lowest order model (based on a single basis function, a constant) in a hierarchy of models that can be derived from a weighted residual procedure.

Despite its simplicity, the Prinja–Pomraning model has limited accuracy in general and does not yield sufficiently accurate transmission probabilities when used for long ducts. This motivated Larsen et al. (1986) to develop an improved 1-D model based on two basis functions, which is expressed as a transport equation that is similar to the usual two-group neutron transport equation, but with angularly dependent cross sections and a "total cross section matrix" that is full, not just diagonal as in the neutron case. Later on, Garcia et al. (2000) extended the work of Larsen et al. (1986) and developed a 1-D model based on three basis functions. Concerning solution methods, both of these works (Larsen et al., 1986; Garcia et al., 2000) made use of a numerical version of the discrete-ordinates method that involves discretization of the spatial variable and convergence by source iteration.

The accuracy of reflection and transmission probabilities computed with the three 1-D models just described has been evaluated for the case of a duct of circular cross section subject to an isotropic incidence of particles, considering wall reflection probabilities between 0.1 and 1.0. The findings of that study (Garcia et al., 2000) are summarized in Table 1, where the parameters Z and ρ denote, respectively, the length and the radius of the duct. The reported deviations are evaluated with respect to results from Monte Carlo simulations of the full 3-D model. It can be concluded from the data in this table that the models based on two and three basis functions were very successful in reducing the deviations of the model based on one basis function. With regard to computational efficiency, a gain that sometimes exceeded one order of magnitude over the

E-mail address: rdgarcia@ieav.cta.br

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Table 1
Maximum percent deviations of approximate 1-D models with respect to the full 3-D model.

Probability	Z/ρ	Number of basis functions		
		One	Two	Three
Reflection	0.1	−15.4	9.1	−5.0
	1.0	−32.8	3.6	−0.3
	10.0	−29.2	2.2	−0.3
Transmission	0.1	0.8	−0.4	0.3
	1.0	29.0	−1.3	−0.3
	10.0	304.6	0.9	−0.2

efficiency of the Monte Carlo method was reported for these models (Larsen et al., 1986; Garcia et al., 2000).

In a recent work (Garcia, 2014), the analytical discrete-ordinates (ADO) method (Barichello and Siewert, 1999) was used to develop a particularly efficient solution for the Prinja–Pomraning model. The main advantages of the ADO method are that the space variable is not discretized and no iteration is required. This, together with a transformation of the angular variable and a good choice of the quadrature scheme, was instrumental in reducing drastically the order of the quadrature needed for a given precision, when compared to the numerical discrete-ordinates method. As a result, a gain of typically one order of magnitude in computational efficiency was achieved. Thus, in this work, encouraged by the excellent performance of the ADO method for the Prinja–Pomraning model, we extend the ADO method for the improved 1-D model with two basis functions of Larsen et al. (1986).

With regard to applications, removal of neutral particles from the confined plasma in tokamaks was the motivation behind most of the early works on approximate 1-D models for particle transport in ducts (Prinja and Pomraning, 1984; Larsen et al., 1986; Malvagi and Pomraning, 1987; Garcia and Ono, 1999; Garcia et al., 2000). A work that allows particle migration in the duct wall (Prinja, 1996) suggested a way for treating the case of penetrating radiation (neutrons and gamma rays). Recently, the approximate 1-D models with one and two basis functions have been applied to non-nuclear topics, such as radiation transport in light ducts (Williams, 2007) and acoustics of long spaces (Jing et al., 2010; Jing and Xiang, 2010).

2. Formulation of the problem

We consider in this work the problem of an evacuated, straight duct of length Z and uniform cross-sectional shape with area A and perimeter L (Larsen et al., 1986). Incoming particles enter the duct by one of its ends (located at $z = 0$) and stream freely until they collide with the duct wall or leave the duct by the other end (located at $z = Z$). Upon collision with the wall, a particle can either be lost from the system by absorption or be diffusely reflected back to the duct interior. Particles that survive after one or more wall collisions will eventually escape the duct by one of its ends.

For this problem, the particle transport equation reduces to

$$\mathbf{\Omega} \cdot \nabla \Psi(\mathbf{r}, \mathbf{\Omega}) = 0, \tag{1}$$

where the angular flux $\Psi(\mathbf{r}, \mathbf{\Omega})$ is a function of the position vector $\mathbf{r} = (x, y, z)$ and the unit vector

$$\mathbf{\Omega} = \left[(1-\mu^2)^{1/2}\cos\varphi, (1-\mu^2)^{1/2}\sin\varphi, \mu\right] \tag{2}$$

that gives the direction of particle travel. The direction $\mathbf{\Omega}$ is determined by two angular variables: $\mu \in [-1, 1]$, the cosine of the polar angle, and $\varphi \in [0, 2\pi]$, the azimuthal angle. The region in space where Eq. (1) is valid corresponds to the interior of the duct and is specified by $R = \{(x, y) | h(x, y) < 0\}$ and $z \in (0, Z)$. Here, the function $h(x, y)$ is used to describe the cross-sectional shape of the duct; for example, for a duct of circular cross section with radius $\rho, h(x, y) = x^2 + y^2 - \rho^2$.

Equation (1) can be written in a more explicit way as

$$(1-\mu^2)^{1/2}\cos\varphi \frac{\partial}{\partial x}\Psi + (1-\mu^2)^{1/2}\sin\varphi \frac{\partial}{\partial y}\Psi + \mu\frac{\partial}{\partial z}\Psi = 0, \tag{3}$$

where $\Psi = \Psi(x, y, z, \mu, \varphi)$. It must be solved subject to boundary conditions given by a specified particle distribution $f(x, y, \mu, \varphi)$ incident at the $z = 0$ duct end and no incoming particles at the $z = Z$ duct end, i.e.

$$\Psi(x, y, 0, \mu, \varphi) = f(x, y, \mu, \varphi) \tag{4a}$$

and

$$\Psi(x, y, Z, -\mu, \varphi) = 0, \tag{4b}$$

for $(x, y) \in R, \mu \in (0, 1]$, and $\varphi \in [0, 2\pi]$. In addition, the solution must satisfy the wall boundary condition (Larsen et al., 1986)

$$-\mathbf{\Omega} \cdot \mathbf{n}\Psi(x, y, z, \mathbf{\Omega}) = \int_{\mathbf{\Omega}' \cdot \mathbf{n} > 0} p(\mathbf{\Omega}' \to \mathbf{\Omega})\Psi(x, y, z, \mathbf{\Omega}')d\Omega', \tag{5}$$

for $(x, y) \in \partial R$, where $\partial R = \{(x, y) | h(x, y) = 0\}$ is the closed curve that describes the contour of the duct wall, $z \in (0, Z)$, and $\mathbf{\Omega} \cdot \mathbf{n} < 0$, where $\mathbf{n}$ denotes the unit normal vector pointing outwards from the duct wall. As Larsen et al. (1986), we consider isotropic wall reflection, and so we take

$$p(\mathbf{\Omega}' \to \mathbf{\Omega}) = -\left(\frac{c}{\pi}\right)(\mathbf{\Omega} \cdot \mathbf{n})(\mathbf{\Omega}' \cdot \mathbf{n}), \tag{6}$$

where $c \in (0, 1]$ is the probability that a particle be reflected towards the duct interior upon collision with the wall.

At this point, we adopt the approximate representation (Larsen et al., 1986)

$$\Psi(x, y, z, \mu, \varphi) \approx \Psi_1(z, \mu)\alpha_1(x, y, \varphi) + \Psi_2(z, \mu)\alpha_2(x, y, \varphi), \tag{7}$$

which expresses the angular flux in terms of the basis functions

$$\alpha_1(x, y, \varphi) = 1 \tag{8a}$$

and

$$\alpha_2(x, y, \varphi) = u[D(x, y, \boldsymbol{\omega}) - v], \tag{8b}$$

where $D(x, y, \boldsymbol{\omega})$ is defined as the distance from a point (x, y) in the x–y plane to the duct wall along the direction $-\boldsymbol{\omega}$, with $\boldsymbol{\omega} = (\cos\varphi, \sin\varphi, 0)$, and the constants u and v are defined as (Larsen et al., 1986)

$$u = \left\{\frac{1}{2\pi A}\int_R \int_0^{2\pi} [D(x, y, \boldsymbol{\omega}) - v]^2 d\varphi dx dy\right\}^{-1/2} \tag{9a}$$

and

$$v = \frac{1}{2\pi A}\int_R \int_0^{2\pi} D(x, y, \boldsymbol{\omega}) d\varphi dx dy. \tag{9b}$$

To determine the unknown coefficients $\Psi_i(z, \mu), i = 1$ and 2, in Eq. (7), we begin by substituting Eq. (7) into Eq. (3) and following the Galerkin prescription of the weighted residual procedure proposed by Larsen (1984) and developed in detail by Larsen et al. (1986), to find that the original problem can be approximated by the problem of solving

$$\mu\frac{\partial}{\partial z}\mathbf{\Psi}(z, \mu) + (1-\mu^2)^{1/2}\mathbf{A}\mathbf{\Psi}(z, \mu) = \frac{2c}{\pi}(1-\mu^2)^{1/2}\mathbf{B}\int_{-1}^{1}(1-\mu'^2)^{1/2}\mathbf{\Psi}(z, \mu')d\mu', \tag{10}$$