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Closed-form solutions for nodal formulations of two dimensional transport problems in heterogeneous media

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ABSTRACT

We present a study of two dimensional, fixed-source neutron transport problems on a heterogeneous medium. It is an extension of a methodology used for solving neutron transport problems on an homogeneous medium, that combines nodal schemes with explicit solutions of the transversal integrated equations via the Analytical Discrete Ordinates method (ADO). We consider closed-form solutions of the integrated discrete ordinates transport equation on a two dimensional cartesian geometry, together with a level symmetric quadrature scheme, on each region of interest, in the domain, possibly characterised by different materials. In this work, each solution in a region is coupled with that of its neighbouring regions to provide the whole solution, without resorting to using iterative schemes. The terms involving unknown angular fluxes that arise using nodal schemes, assumed as constant function approximations, are added to the source term. The model proposed leads to a considerable reduction of the order of the associated eigenvalue problems written as perturbations of diagonal matrices, and the solutions obtained are explicit in terms of the spatial variables. Analytical expressions are also derived for the elementary solutions. The numerical results obtained are shown to be in good agreement with other results available in the literature.

1. Introduction

The study and development of analytical techniques is an important line of research in the field of multidimensional particle transport theory (Williams, 2013), in order to search for more efficient computational solutions or to establish benchmark results (Ganapol et al., 2013). In the solution of such transport problems, nodal schemes (Dorning, 1979; Wagner, 1979; Walters and O'Dell, 1981; Badruzzaman, 1985; Lawrence, 1986; Azmy, 1988; Barros and Larsen, 1992; Mello and Barros, 2002; Poursalehi et al., 2013; Menezes et al., 2014; Altac and Tekkalmaz, 2014) are used to reduce the complexity of the associated models, as they are more amenable to the use of analytical techniques.

In this work, we extend the application of a methodology that combines nodal schemes with ADO method (*Analytical Discrete Ordinates*) (Barichello and Siewert, 1999) for two-dimensional neutron transport problems in heterogeneous media. This method has been successfully used to solve two dimensional particle transport models (Barichello et al., 2011; Picoloto et al., 2013; Prolo Filho and Barichello, 2013; Tres et al., 2014) and different approaches have been taken to specify auxiliary equations that are needed to represent the unknown border fluxes which arise when one uses nodal schemes, like approximation by constants (Prolo Filho and Barichello, 2013; Tres et al., 2014), by combinations of certain exponential functions (Prolo Filho, 2011; Prolo Filho and Barichello, 2014) and by equations specifying relationships between the unknown outgoing fluxes with the integrated, average fluxes (Barichello et al., 2011; Picoloto et al., 2013; Barichello et al., 2015). Except by Barichello et al. (2015), all problems treated previously consider an homogeneous medium.

In this work we have used a different approach to study a medium composed by different materials. The domain was divided in several regions where an explicit solution is derived for each region and, unlike previous work (Barichello et al., in press), the unknown angular fluxes on boundaries and interfaces are considered as part of the source term. In the general problem, the regions are coupled and we do not use any iterative processes either to relate the different regions or to obtain the solution of the general linear system which estimate the unknown fluxes in the interior and boundaries of the domain.

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We will proceed as follows. In the next section, the two dimensional neutron transport equation in discrete ordinates form is introduced and nodal methods are applied to it, obtaining one dimensional transverse integrated equations in each region r. In Section 3 we apply the ADO method to the one dimensional equations obtaining eigenvalue problems for the x and y directions in a region r; we also derive analytical elementary solutions, proposing a particular solution and the approximations for the unknown fluxes on the borders therefore establishing the general solution. Test problems and the numerical results are presented in Section 5. We conclude by presenting our final remarks in Section 6.

2. Formulation

We begin with the two dimensional neutron transport equation, time-independent, on a non-multiplicative medium, with isotropic scattering, one energy group, on a rectangular domain D, with $x \in [0, a]$ and $y \in [0, b]$, written in discrete ordinates form as (Lewis and Miller, 1984)

$$\mu_i \frac{\partial}{\partial x}\Psi(x,y,\mathbf{\Omega}_i) + \eta_i \frac{\partial}{\partial y}\Psi(x,y,\mathbf{\Omega}_i) + \sigma_t\Psi(x,y,\mathbf{\Omega}_i) = Q(x,y) + \frac{\sigma_s}{4}\sum_{k=1}^{M} w_k \Psi(x,y,\mathbf{\Omega}_k) \tag{1}$$

for $i = 1,\ldots,M$, with $M = N(N+2)/2$, according to the level symmetric quadrature scheme, where N refers to the approximation order S_N; w_i are the weights associated to the directions $\mathbf{\Omega}_i = (\mu_i, \eta_i)$; $Q(x,y)$ is the isotropic neutron source term; and σ_t and σ_s are the total and scattering macroscopic cross sections, respectively.

As we are interested in investigating the treatment of problems where the medium may be made of different materials, we consider the domain subdivided in r regions defined as $x \in [a_{r-1}, a_r]$ and $y \in [b_{r-1}, b_r]$, with $a_0 = 0 < \cdots < a_R = a$ and $b_0 = 0 < \cdots < b_R = b$.

We used as a basis the explicit solutions derived for those transport problems in homogeneous media. Following (Barichello et al., 2011), we established an ordering on the directions $\mathbf{\Omega}_i$ such that for indices $i = 1,\ldots,M/2$ the directions have coordinate $\mu_i > 0$; and $\mu_i < 0$ for indices $i = M/2+1,\ldots,M$. Therefore, using nodal method techniques to obtain the one dimensional transverse integrated equations in direction x on a region r, we integrate Eq. (1) for every $y \in [b_{r-1}, b_r]$, obtaining

$$\mu_i \frac{d}{dx}\Psi_{yr}(x,\mathbf{\Omega}_i) + \sigma_{tr}\Psi_{yr}(x,\mathbf{\Omega}_i) = Q_{yr}(x,\Omega_i) + \frac{\sigma_{sr}}{4}\sum_{k=1}^{M/2} w_k\left[\Psi_{yr}(x,\mathbf{\Omega}_k) + \Psi_{yr}(x,\mathbf{\Omega}_{k+M/2})\right] \tag{2}$$

and

$$-\mu_i \frac{d}{dx}\Psi_{yr}(x,\mathbf{\Omega}_{i+M/2}) + \sigma_{tr}\Psi_{yr}(x,\mathbf{\Omega}_{i+M/2}) = Q_{yr}(x,\Omega_{i+M/2}) + \frac{\sigma_{sr}}{4}\sum_{k=1}^{M/2} w_k\left[\Psi_{yr}(x,\mathbf{\Omega}_k) + \Psi_{yr}(x,\mathbf{\Omega}_{k+M/2})\right], \tag{3}$$

for $i = 1,\ldots,M/2$. In these equations,

$$\Psi_{yr}(x,\mathbf{\Omega}_i) = \frac{1}{\alpha_r}\int_{b_{r-1}}^{b_r}\Psi_r(x,y,\mathbf{\Omega}_i)dy, \tag{4}$$

$$Q_{yr}(x,\mathbf{\Omega}_i) = Q_{yr}(x) - \frac{\eta_i}{\alpha_r}\left[\Psi_r(x,b_r,\mathbf{\Omega}_i) - \Psi_r(x,b_{r-1},\mathbf{\Omega}_i)\right] \tag{5}$$

and

$$Q_{yr}(x) = \frac{1}{\alpha_r}\int_{b_{r-1}}^{b_r} Q_r(x,y)dy, \tag{6}$$

where $\alpha_r = b_r - b_{r-1}$ and $i = 1,\ldots,M$. In the derivation presented here, we used $\eta_i = \eta_{i+M/2}$, for $i = 1,\ldots,M/2$, according to the level symmetric quadrature (Lewis and Miller, 1984).

Once obtained the system of one dimensional differential equations dependent on x in region r, we proceed similarly to obtain such a system in terms of the variable y. Again, following previous work (Barichello et al., 2011), we associate indices $i = 1,\ldots,M/2$ to directions with coordinate $\eta_i > 0$ and indices $i = M/2+1,\ldots,M$ to directions with coordinates $\eta_i < 0$. Integrating Eq. (1) for every $x \in [a_{r-1}, a_r]$, we obtain

$$\eta_i \frac{d}{dy}\Psi_{xr}(y,\mathbf{\Omega}_i) + \sigma_{tr}\Psi_{xr}(y,\mathbf{\Omega}_i) = Q_{xr}(y,\Omega_i) + \frac{\sigma_{sr}}{4}\sum_{k=1}^{M/2} w_k\left[\Psi_{xr}(y,\mathbf{\Omega}_k) + \Psi_{xr}(y,\mathbf{\Omega}_{k+M/2})\right] \tag{7}$$

and

$$-\eta_i \frac{d}{dy}\Psi_{xr}(y,\mathbf{\Omega}_{i+M/2}) + \sigma_{tr}\Psi_{xr}(y,\mathbf{\Omega}_{i+M/2}) = Q_{xr}(y,\Omega_{i+M/2}) + \frac{\sigma_{sr}}{4}\sum_{k=1}^{M/2} w_k\left[\Psi_{xr}(y,\mathbf{\Omega}_k) + \Psi_{xr}(y,\mathbf{\Omega}_{k+M/2})\right], \tag{8}$$

for $i = 1,\ldots,M/2$. In these equations, we define

$$\Psi_{xr}(y,\mathbf{\Omega}_i) = \frac{1}{\beta_r}\int_{a_{r-1}}^{a_r}\Psi_r(x,y,\mathbf{\Omega}_i)dx \tag{9}$$

and the source terms

$$Q_{xr}(y,\mathbf{\Omega}_i) = Q_{xr}(y) - \frac{\mu_i}{\beta_r}\left[\Psi_r(a_r,y,\mathbf{\Omega}_i) - \Psi_r(a_{r-1},y,\mathbf{\Omega}_i)\right], \tag{10}$$

$$Q_{xr}(y) = \frac{1}{\beta_r}\int_{a_{r-1}}^{a_r} Q_r(x,y)dx, \tag{11}$$

where $i = 1,\ldots,M$, r indicates a region, $x \in [a_{r-1}, a_r]$, with $a_0 = 0 < \cdots < a_R = a$, $\beta_r = a_r - a_{r-1}$ and, following the level symmetric quadrature (Lewis and Miller, 1984), we have used the relation $\mu_i = \mu_{i+M/2}$.

We note that on Eqs. (5) and (10) we have incorporated to the source term the angular fluxes on the border of each region, arising from the integration of Eq. (1). Some of these variables may be known from the boundary conditions established for the domain, whereas auxiliary equations will be introduced for the others, as is usual on nodal schemes. In the sequel we will specify the non-homogeneous term and the boundary conditions for the test problems, and will discuss and define the unknowns present on the right-hand-side of Eqs. (5) and (10).

3. Solution by the ADO method in a region *r*

The use of nodal schemes in the two dimensional transport equation allowed us to obtain a system of one dimensional equations integrated transversely which will be solved using the ADO method. For a region r, we propose solutions of the homogeneous problem, for $i = 1,\ldots,M$, as

$$\Psi^H_{yr}(x,\mathbf{\Omega}_i) = \Phi_{yr}(\nu_r,\mathbf{\Omega}_i)e^{-x/\nu_r}. \tag{12}$$

Substituting Eq. (12) in Eqs. (2) and (3), we obtain

$$-\frac{\mu_i}{\nu_r}\Phi_{yr}(\nu_r,\mathbf{\Omega}_i) + \sigma_{tr}\Phi_{yr}(\nu_r,\mathbf{\Omega}_i) = \frac{\sigma_{sr}}{4}\sum_{k=1}^{M/2} w_k\left[\Phi_{yr}(\nu_r,\mathbf{\Omega}_k) + \Phi_{yr}(\nu_r,\mathbf{\Omega}_{k+M/2})\right] \tag{13}$$

and

$$\frac{\mu_i}{\nu_r}\Phi_{yr}(\nu_r,\mathbf{\Omega}_{i+M/2}) + \sigma_{tr}\Phi_{yr}(\nu_r,\mathbf{\Omega}_{i+M/2}) = \frac{\sigma_{sr}}{4}\sum_{k=1}^{M/2} w_k\left[\Phi_{yr}(\nu_r,\mathbf{\Omega}_k) + \Phi_{yr}(\nu_r,\mathbf{\Omega}_{k+M/2})\right], \tag{14}$$