Annals of Nuclear Energy 86 (2015) 72–79

Contents lists available at ScienceDirect

Annals of Nuclear Energy

journal homepage: www.elsevier.com/locate/anucene

Angular and spatial moments for half-space albedo transport problems

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ARTICLE INFO

Article history:
Received 1 September 2014
Received in revised form 20 December 2014
Accepted 22 December 2014

In memory of Ivan Kuščer 1918–2000 and dedicated to M. M. R. Williams.

Keywords:
Radiative transfer
Neutron transport
Angular moments
Spatial moments
Case's method
Chandrasekhar *H*-function

ABSTRACT

Two classic problems of radiative transfer and neutron transport are solved for a spatially-uniform semi-infinite medium with isotropic scattering. General analytical equations are derived for (1) angular moments of the outward current and (2) spatial moments of the total flux within the half-space. Such moments, for example, can provide analytically explicit equations for determining the surface albedo of the medium as well as the mean depth and mean square distance of travel within the medium. The analysis is done with the Case-style eigenmode method as expressed in terms of the Chandrasekhar *H*-function and its moments.

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1. Introduction

We consider the elementary imaging problem of light emerging from a one-dimensional half-space due to uniform illumination over its surface. This problem was elegantly treated in an analytical manner by Chandrasekhar (1960) who obtained the emerging angular distribution in terms of his classic *H*-function and a few of its angular moments. In the 1960s the eigenmode expansion technique (Case and Zweifel, 1967) was developed that enabled more easily the analytical determination of the light field within, as well as emerging from, the half-space.

The eigenmode expansion technique is used here to solve for the angular moments of the outward angular current at the surface and for the spatial moments of the total flux within the half-space for two general incident illumination boundary conditions from which even more general boundary conditions can be constructed. The derivation here illustrates the elegance of employing the eigenmode expansion technique with half-range orthogonality relations for isotropic scattering that contain the weight function $\mu H(\mu)$, rather than the weight function used in their original derivation (Case and Zweifel, 1967; Kuščer et al., 1964).

Moment-driven approximations have a long history in applied transport theory. The explicit expressions for angular and spatial moments in a semi-infinite medium derived here, with the complication of leakage from the medium-vacuum interface of the half-space, are analogous to those for a localized plane source in an infinite medium with isotropic scattering developed by Ganapol (1985). The infinite medium problem with lethargy-dependence and anisotropic scattering was solved by Cacuci and Goldstein (1977). Larsen (1996) has developed a moments approach to solving infinite medium 1D, 2D, and 3D problems that includes energy and time dependence; see also an extensive discussion of the moments method for such problems by Sanchez (2003).

Recent examples of the use of analytic moments is illustrated in the verification and validation of energy moments for diffusion theory calculations (Crawford and Ring, 2012a; Crawford and Ring, 2012b). When applying approximate analytic searchlight solutions based on the method of images, some limitations of diffusion theory can be avoided by configuring positive and negative source configurations using known properties of the corresponding 1D problem (d'Eon, 2014a). The nonlinear optimization problem in such an approach potentially can be avoided by solving for a source configuration that satisfies either angular or spatial moments derived here. Additionally, zero-variance based variance-reduction schemes for Monte Carlo methods require efficient and accurate approximate solutions to guide random walks. It is likely that moment-driven approximations for the interior solution in a half space could provide improved accuracy while still permitting analytic sampling for 1D and searchlight problems (Křivánek and d'Eon, 2014b).

Another application of spatial moments is the computation of the mean distance of travel within an idealized, spatially-uniform atmosphere or ocean. The mean depth before a photon is absorbed

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http://dx.doi.org/10.1016/j.anucene.2014.12.021
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or escapes through the surface can provide a measure of the euphotic zone, while the mean square distance of travel gives a measure of the average total distance traveled by a photon before either event occurs (Kirk, 1994; Mobley, 1994).

The angular moments of the emerging intensity and the spatial moments of the interior intensity also provide a different set of analytical equations for performing benchmark accuracy checks of solutions of idealized half-space albedo problems to supplement those of Ganapol (2008). An advantage of using a set of moments for computing the distribution in the interior, rather than an eigenmode expansion of the analytically-derived angular intensity, is that no singular integrals need to be numerically evaluated.

Following a mathematical statement of the objectives of the paper in the next section, the eigenmodes and Chandrasekhar functions are reviewed in Section 3 before deriving the equations for the angular moments of the emerging angular intensity in Section 4. This is followed by equations for obtaining the spatial moments of the total flux in Section 5 and numerical results for both sets of moments in Section 6. For simplicity the treatment of an azimuthally-dependent incident illumination is deferred until Section 7 where the extension to anisotropic scattering also is briefly discussed.

2. Objectives

We restrict our attention to isotropic scattering and consider the one-speed radiative transfer equation (Case and Zweifel, 1967)

$$(\mu\partial/\partial x + 1)I(x,\mu) = (c/2)\int_{-1}^{1} I(x,\mu')\mathrm{d}\mu', \quad x \geqslant 0, \ -1 \leqslant \mu \leqslant 1, \tag{1}$$

where c is the single scattering albedo with $0 \leqslant c < 1$, x is the distance into the medium in mean free paths, and μ is the cosine of the polar angle with respect to the inward surface normal. Both collimated and diffuse boundary conditions

$$I(0,\mu;\mu_\ell) = \delta(\mu - \mu_\ell) \tag{2a}$$

$$I_k(0,\mu) = \mu^k, \quad k = 0, \ 1, \ \ldots, \tag{2b}$$

will be considered for μ and $\mu_\ell \in [0,1]$. Combination of these boundary conditions gives the quite general surface illumination condition

$$I_{in,\,total}(0,\mu) = \sum_{\ell=1}^{L} c_\ell \delta(\mu - \mu_\ell) + \sum_{k=0}^{K} d_k \mu^k. \tag{3}$$

The coefficients of the incident illumination, c_ℓ and d_k, are assumed known and, if normalized such that

$$\sum_{\ell=1}^{L} c_\ell + \sum_{k=0}^{K} d_k/(k+1) = 1, \tag{4}$$

they are the fractions in each of the incident illumination modes, respectively, as defined by Eqs. (2a) and (2b).

2.1. Angular moments of the emerging intensity

Our first objective is to obtain analytical equations for computing the angular moments of the intensity emerging from the half-space,

$$j_{out,n} = \int_0^1 \mu^{n+1} I(0,-\mu)\mathrm{d}\mu, \quad n = 0, \ 1, \ldots, \tag{5a}$$

from knowledge of the known incident monochromatic illumination

$$j_{in,n} = \int_0^1 \mu^{n+1} I(0,\mu)\mathrm{d}\mu, \quad n = 0, \ 1, \ldots. \tag{5b}$$

The emerging moments provide information about the emerging intensity without the need to compute the angular intensity $I(0,-\mu), 0 \leqslant \mu \leqslant 1$. The outward partial current $j_{out,0}$ divided by the inward partial current $j_{in,0}$, for example, is just the albedo of the half-space.

From Eqs. (3), (5a) and (5b), the moments of the total angular emerging and incident distributions then are

$$j_{out,n,\,total} = \sum_{\ell=1}^{L} c_\ell j_{out,n}(\mu_\ell) + \sum_{k=0}^{K} d_k j_{out,n,k} \tag{6a}$$

$$j_{in,n,\,total} = \sum_{\ell=0}^{L} c_\ell \mu_\ell^{n+1} + \sum_{k=1}^{k} d_k (n+k+2)^{-1}, \tag{6b}$$

where

$$j_{out,n}(\mu_\ell) = \int_0^1 \mu^{n+1} I(0,-\mu;\mu_\ell)\mathrm{d}\mu \tag{7a}$$

$$j_{out,n,k} = \int_0^1 \mu^{n+1} I_k(0,-\mu)\mathrm{d}\mu \tag{7b}$$

$$j_{in,n}(\mu_\ell) = \int_0^1 \mu^{n+1} I(0,\mu;\mu_\ell)\mathrm{d}\mu \tag{7c}$$

$$j_{in,n,k} = \int_0^1 \mu^{n+1} I_k(0,\mu)\mathrm{d}\mu. \tag{7d}$$

After $j_{out,n,\,total}$ and $j_{in,n,\,total}$ have been determined for a prescribed set of c_ℓ and d_k values, we can obtain the results for

$$\beta_n = j_{out,n,total}/j_{in,0,total}, \tag{8}$$

which are the fractions of the incident current (i.e., the 0th angular mode) that emerge in the nth angular mode, with β_0 the surface albedo (i.e., the ratio of the outward to the inward current) of the semi-infinite medium.

When computing β_n values it is numerically more convenient, however, to separately obtain analytical equations for each boundary condition by computing the normalized ratios

$$j_{ratio,n}(\mu_\ell) = j_{out,n}(\mu_\ell)/j_{in,0}(\mu_\ell) \tag{9a}$$

$$j_{ratio,n,k} = j_{out,n,k}/j_{in,0,k} \tag{9b}$$

that lie in the range $[0,1]$. The ratios $j_{ratio,n}(\mu_\ell)$ and $j_{ratio,n,k}$ are the fractions of the incident current that emerge from the surface in the nth angular current mode for the incident illumination conditions of Eqs. (2a) and (2b), with $j_{ratio,0}(\mu_\ell)$ and $j_{ratio,0,k}$ the surface albedo for each boundary condition.

With $j_{in,0}(\mu_\ell) = \mu_\ell$ and $j_{in,0,k} = (k+2)^{-1}$, instead of employing Eq. (6a) we evaluate $j_{out,n,total}$ with the ratios $j_{ratio,n}(\mu_\ell)$ and $j_{ratio,n,k}$,

$$j_{out,n,\,total} = \sum_{\ell=1}^{L} c_\ell \mu_\ell j_{ratio,n}(\mu_\ell) + \sum_{k=0}^{K} d_k (k+2)^{-1} j_{ratio,n,k}, \tag{10}$$

while using Eq. (6b) for $j_{in,n,total}$.

2.2. Spatial moments of the interior flux

Our second objective is to obtain the spatial moments of the interior intensity as given by

$$\rho_n = \int_0^\infty x^n \rho(x)\mathrm{d}x, \quad n = 0, \ 1, \ldots, \tag{11a}$$

where the interior flux is

$$\rho(x) = \int_{-1}^{1} I(x,\mu)\mathrm{d}\mu. \tag{11b}$$

The ρ_n are moments of the distribution with respect to the penetration distance. Such moments provide information about the depth of penetration of the incident illumination that can serve as a

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