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Closed-form solution of the first-order Transport-Driven Diffusion approximation

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ABSTRACT

The paper presents a closed-form solution of the first-order Transport-Driven Diffusion (TDD) proposed by Picca and Furfaro (2014). The solution is rigorously derived for two source configurations (i.e. Dirac's delta and rectangular source) to explicitly describe the contribution of the uncollided component and the collided diffuse term. The results are employed to compare the solution of first-order Transport-Driven Diffusion (TDD_T with $T = 1$) with the analytical diffusion approximation as well as to validate numerical solution of the TDD_1. The comparison with higher-order TDD (i.e., (TDD_T with $T > 1$) is also reported.

1. Introduction

Besides its beauty and elegance, the analytical solution of a physico-mathematical problem is of general interest because it can provide insights on how input parameters may influence the solution. Additionally, analytical results can be useful for deriving benchmark reference solutions not affected by *numerical errors*. Such reference solutions are generally used for code validation.

The closed-form solution of linear transport equation is possible only for a very limited number of physical problems. For instance, for certain configurations it is possible to derive a fully analytical solution of the diffusion approximation for the Linear Boltzmann Equation (LBE). Although not affected by *numerical errors*, the analytical diffusion solution inherently suffers from *model errors* and hence provides a distorted description for the real physical problem of neutral particle transport. For instance, it is well-known that the approximation of LBE with the diffusion equation is not particularly accurate in a low scattering medium with localized sources (e.g., Davison, 1957).

This paper explores the possibility of limiting the diffusion error without losing the advantages of a closed-form solution. The starting point for this quest is a novel approach for the solution of LBE proposed by Picca and Furfaro (2014). The so-called Transport-Driven Diffusion (TDD) is a hybrid methodology which combines transport theory with diffusion theory for an optimized description of source-driven problems, typically dominated by transport for the first collisions and by diffusive collective behavior after multiple collisions. As described in Picca and Furfaro (2014), the flexibility of the method lies in the possibility to set *ad hoc* the transition between transport and diffusion, TDD_T being the approximation where after the first T collisions the diffusion model is used.

In general, the above-mentioned TDD methodology needs numerical tools for its solution. Conversely, the present work seeks a closed-form solution of simplified configurations using the first-order TDD (i.e., TDD_1). For this purpose, a slab geometry homogeneous medium is considered and the analytical developments are pushed as far as possible, using numerical tools with numerical error control only where strictly necessary. The paper is organized as follows: In Section 2, the neutral particle equation is presented. In Section 3, the solution for diffusion and TDD_1 approximation for the problem driven by Dirac's delta source term at the center of the system is derived and results compared with a higher order TDD approximation as well as with numerical discrete ordinate solution (both of them solved numerically), which are regarded as reference lines for comparison of lower order approximations (i.e., diffusion and TDD_1). Section 4 considers a problem driven by a symmetric rectangular source and presents the corresponding diffusion and TDD_1 approximations. Section 5 presents the conclusions.

2. Neutral particle transport equation

The steady-state LBE in its one-dimension and one-angle formulation for an homogeneous medium with isotropic scattering can be written as (Davison, 1957):

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$$\left[\mu\frac{\partial}{\partial x}+\sum\right]\varphi(x,\mu)=\frac{\Sigma_s}{2}\int_{-1}^{+1}\phi(x,\mu)d\mu+\frac{1}{2}s(x), \tag{1}$$

where μ is the propagation direction (measured with respect to positive x-axis), Σ the total cross section (inverse of absorption mean free path), Σ_s the scattering cross section (inverse of scattering mean free path), $\psi(x,\ \mu,\ t)$ the angular flux (total length traveled by neutrons in direction μ per unit time and volume), $\int_{-1}^{+1}\psi(x,\mu,t)d\mu$ scalar flux and $s(x)$ the source term.

In case of homogeneous boundary conditions, the vacuum conditions are set on entering angular flux, i.e.:

$$\begin{aligned}&\phi(-H/2,\ \mu>0)=0\\&\phi(+H/2,\ \mu<0)=0\end{aligned} \tag{2}$$

The problem defined in Eqs. (1) and (2) can be approximated with the following diffusion equation (Davison, 1957):

$$\left[-D\frac{d^2}{dx^2}+\Sigma_r\right]\phi(x)=s(x) \tag{3}$$

and boundary condition:

$$\phi(\pm H/2)=0 \tag{4a}$$

where

$$\Sigma_r=\Sigma-\Sigma_s \tag{4b}$$

$$D=\frac{1}{3\sum} \tag{4c}$$

The source terms in Eqs. (1) and (3) differ by the multiplication factor 1/2 which is the angular redistribution in the one-angle, one-dimensional geometry, which applied to transport and not to diffusion.

In alternative to diffusion theory, the hybrid methodology proposed by Picca and Furfaro (2014) can be applied to approximate transport equation. The basic idea behind the first-order Transport-Driven Diffusion (i.e., TDD_1) is to seek the solution as a sum of a collided (i.e. ϕ_c) and uncollided components (i.e., ϕ_u), i.e.:

$$\phi(x)=\int_{-1}^{+1}\phi_u(x,\mu)d\mu+\phi_c(x) \tag{5}$$

The models employed for each of these components are the following collisionless transport equation and the diffusion equation, respectively:

$$\left[\mu\frac{\partial}{\partial x}+\sum\right]\phi_u(x,\mu)=\frac{1}{2}s_u(x) \tag{6}$$

and:

$$\left[-D\frac{d^2}{dx^2}+\Sigma_r\right]\phi_c(x)=s_c(x) \tag{7a}$$

where $s_u(x)=s(x)$:

$$s_c(x)=\int_{-1}^{+1}\frac{\Sigma_s}{2}\int_{-1}^{+1}\phi_u(x,\mu)d\mu d\mu'=\Sigma_s\int_{-1}^{+1}\phi_u(x,\mu)d\mu \tag{7b}$$

The following Sections details the solution of Eq. (3) for diffusion and Eqs. (6) and (7) for TDD_1 for two source types.

3. Closed-form approximations of linear transport solution: response to a Dirac's delta function

In this section, the solution for diffusion and TDD_1 approximation for the problem driven by Dirac's delta source term at the center of the system is derived.

3.1. Diffusion approximation

In a homogeneous medium, the solution of the diffusion equation can be derived by means of eigenfunction expansion of the Laplacian in Eq. (3). For the slab geometry, the Helmholtz problem can be written as follows (Hildebrand, 1965):

$$\frac{d^2}{dx^2}\varphi=-B^2\varphi \tag{8}$$

and, considering the b.c. in Eq. (4), the normalized eigenfunctions are:

$$\phi_n(x)=\sqrt{\frac{2}{H}}\cos[B_nx] \tag{9a}$$

where

$$B_n=\frac{(2n-1)}{H/2}\frac{\pi}{2}=\frac{(2n-1)\pi}{H} \tag{9b}$$

For $n=1,\ \ldots,\ \infty$. Using the orthogonality property of the eigenfunctions in Eq. (9a): $\int_{-H/2}^{+H/2}\varphi_n(x)\varphi_m(x)dx=\delta_{nm}$, the solution can be written as:

$$\phi(x)=\Sigma_{n=1}^{\infty}C_n\varphi_n(x) \tag{10a}$$

where

$$C_n=\frac{S_n}{\Sigma_r+DB_n^2} \tag{10b}$$

The projection of the source onto the eigenfunctions reads as follows:

$$S_n=\int_{-H/2}^{+H/2}\varphi_n(x)s(x)dx \tag{11}$$

When considering a Dirac's delta source term in the middle of the system (i.e., $s(x)=\delta(x)$), it is easy to prove that the coefficients in Eq. (9) become:

$$S_n=\sqrt{\frac{2}{H}} \tag{12}$$

which is a constant with respect to n.

3.2. First-order transport-driven approximation

For the problem under analysis, the driving force in Eq. (7a) becomes $s_u(x)=\delta(x)$ and it can be proven that the uncollided component for $\mu>0$ is Picca and Furfaro (2014):

$$\phi_u(x,\mu>0)=\begin{cases}\frac{1}{2}\frac{e^{-\frac{\Sigma x}{\mu}}}{\mu} & \text{for } x>0\\ 0 & \text{for } x<0\end{cases} \tag{13a}$$

and, considering the symmetries in the problem when simultaneously changing $x\to-x$ and $\mu\to-\mu$, for $\mu<0$ it reads:

$$\phi_u(x,\mu<0)=\begin{cases}0 & \text{for }\ x>0\\ -\frac{1}{2}\frac{e^{-\frac{\Sigma x}{\mu}}}{\mu} & \text{for }\ x<0\end{cases} \tag{13b}$$

Using the identity: $-\int_{-1}^{0}\frac{e^{-\frac{\Sigma x}{\mu}}}{\mu}d\mu=\int_{-1}^{0}\frac{e^{+\frac{\Sigma x}{|\mu|}}}{|\mu|}d\mu=\int_{0}^{+1}\frac{e^{+\frac{\Sigma x}{\mu}}}{\mu}d\mu$ and the definition of exponential integral $E_1(\xi)=\int_0^1\frac{e^{-\xi/z}}{z}dz$, the scalar flux can be written as:

$$\int_{-1}^{+1}\phi_u(x,\mu)d\mu=\begin{cases}\frac{1}{2}\int_0^{+1}\frac{e^{-\frac{\Sigma x}{\mu}}}{\mu}d\mu & \text{for } x>0\\ -\frac{1}{2}\int_{-1}^{0}\frac{e^{-\frac{\Sigma x}{\mu}}}{\mu}d\mu & \text{for } x<0\end{cases}=\frac{1}{2}E_1(\Sigma|x|) \tag{14}$$

and the source term for diffusion in Eq. (7b) as $s_c(x)=\frac{1}{2}\Sigma_sE_1(\Sigma|x|)$. The solution of diffusion model can be determined as detailed in