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Numerical results for the transport equation with strongly anisotropic scattering in a slab

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ABSTRACT

In this work we solve the transport equation with strongly anisotropic scattering, i.e., with a forward–backward-anisotropic kernel. We treat the problem by finding an integral representation to the solution, which we then project to a finite dimensional space. We verify numerically the robustness of the techniques we develop by performing calculations for several cases found in the literature. Then we obtain new numerical results for the transport equation with strongly anisotropic scattering when the kernel has more than two terms. Our simulations allows us to obtain the total intensity, the total flux, the dominant eigenvalue and the critical thickness with precision to at least five digits. These simulations indicate that adding third and fourth kernel terms contributes to about 1% in the studied cases.

1. Introduction

The transport equation with forward and backward scattering was first introduced by Fermi (see Williams, 1985) and later extended by Williams (1966), Inönü (1973) and Sahni et al. (1992) providing greater flexibility in representing realistic scattering cross sections (see the comments of Davison and Sykes, 1958). Although, these kernels were introduced for modeling of neutron transport, they have also been used in radiative transfer problems.

Much effort has been made to study the transport equation with backward–forward-isotropic and backward–forward-linearly anisotropic scattering in the last years. For instance, Spiga and Vestrucci (1981) developed a semi-analytical algorithm to calculate the critical total flux of the transport operator. Sahni et al. (1992) applied discrete ordinates S_N combined with the Carlvik's method to compute the eigenvalues of this operator. Ganapol and Kornreich (1996) simulated the scalar flux by their Green's function method. Anli (2001) used a spectral Green's function method and the diamond difference scheme to obtain the total intensity. Awatif and Elghazaly (2004) applied variational techniques to calculate the albedo while Öztürk (2014) used U_N method to compute the critical thickness.

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Sahni et al. (1992) consider the case of backward–forward-linearly anisotropic scattering, but as observed by the authors themselves the inclusion of anisotropic kernels of higher order in their methodology yield great complications, thereby limiting their methodology. Even more recent works (see, for instance, Öztürk, 2014; Yilmazer, 2007; Bülbül et al., 2011) have not presented solutions for the general case where the kernel has very strong anisotropies.

In the present paper we deal with the more general case involving strongly anisotropic scattering kernels using an extension of the method introduced in our previous works (Azevedo et al., 2011, 2013a). The method consists of transforming the transport problem by an integral representation and projecting the involved operators in finite dimensional spaces. Here, we focus our discussion on the numerical point of view, while observing that the error bounds in the present case can certainly be obtained using the ideas presented by Azevedo et al. (2013a) and Sauter et al. (2012).

The neutron transport equation for strongly anisotropic scattering in a slab is given by

$$\mu\frac{\partial I}{\partial y} + \lambda' I = \sigma'\int_{-1}^{1}\omega'(\mu,\mu')I(y,\mu')d\mu' + Q(y), \quad y \in (0,L) \tag{1}$$

$$I(0,\mu) = B_0(\mu), \qquad \mu > 0, \tag{2}$$

$$I(L,\mu) = B_L(\mu), \qquad \mu < 0, \tag{3}$$

where $0 \leqslant y \leqslant L$ is the spatial variable, μ is the cosine of the angle formed between the direction of the propagation and the axis

y, $Q(y)$ is the source term, σ' is a constant and $\omega'(\mu,\mu')$ is the scattering kernel which is given by

$$\omega'(\mu,\mu') = \frac{(1-\alpha'-\beta')}{2}\omega(\mu,\mu') + \alpha'\delta(\mu-\mu') + \beta'\delta(\mu+\mu') \tag{4}$$

where α' and β' are constants, $\alpha'+\beta' \leqslant 1$, δ is a Dirac delta,

$$\omega(\mu,\mu') = \sum_{l=0}^{M} b_l P_l(\mu) P_l(\mu'), \tag{5}$$

β_l are constants and P_l are Legendre Polynomials of order l.

In the Section 2, we rewrite the model (1)–(3) in a integral formulation. In the Section 3, we apply the kernel structure (5) to write the integral formulation in a system of more simple operators. In the Section 4, we discretize the integral operator in finite dimensional matrices. In the Section 5, we validate the methodology and compute numerical results for the transport equation with strongly anisotropic scattering. In the Section 6, we give our final conclusions.

2. Integral formulation

Following the ideas presented by Azevedo et al. (2011, 2013a), Sauter et al. (2012) we reformulate the problem (1)–(3) to obtain the solution in terms of an integral equation. First, we use the kernel ω' (4) and the properties of the Dirac delta function to rewrite Eq. (1) as follows:

$$\mu\frac{\partial I}{\partial y}(y,\mu) + \lambda I(y,\mu) - \beta I(y,-\mu) = \frac{\sigma}{2}\int_{-1}^{1}\omega(\mu,\mu')I(y,\mu')d\mu' + Q(y), \tag{6}$$

with $y \in (0,L)$ and $t > 0$, where $\lambda = \lambda' - \sigma'\alpha'$, $\beta = \sigma'\beta'$, and $\sigma = \sigma'(1-\alpha'-\beta')$.

Now we rewrite the problem (1)–(3) in a integral formulation. For this propose, we consider the auxiliary problem:

$$\mu\frac{\partial I}{\partial y}(y,\mu) + \lambda I(y,\mu) - \beta I(y,-\mu) = q(y,\mu), \tag{7}$$

$$I(\mu) = B_0(\mu), \qquad \mu > 0, \tag{8}$$

$$I(\mu) = B_L(\mu), \qquad \mu < 0. \tag{9}$$

This problem can be solved by applying the method of ray tracing, which consists in integrating the transport equation along the ray's direction (see, for instance, Sauter et al., 2012; Modest, 2003). Eq. (7) is equivalent to the system:

$$\frac{\partial}{\partial y}\begin{bmatrix} I(y,\mu) \\ I(y,-\mu)\end{bmatrix} + E(\mu)\begin{bmatrix} I(y,\mu) \\ I(y,-\mu)\end{bmatrix} = F(y,\mu) \tag{10}$$

where

$$E(\mu) = \frac{1}{\mu}\begin{bmatrix} \lambda & -\beta \\ \beta & -\lambda \end{bmatrix} \quad \text{and} \quad F(y,\mu) = \begin{bmatrix} \frac{1}{\mu}q(y,\mu) \\ -\frac{1}{\mu}q(y,-\mu)\end{bmatrix} \tag{11}$$

The solution of (10) can be expressed in two ways:

$$\begin{bmatrix} I(y,\mu) \\ I(y,-\mu)\end{bmatrix} = e^{-E(\mu)y}\begin{bmatrix} I(0,\mu) \\ I(0,-\mu)\end{bmatrix} + \int_0^y e^{-E(\mu)(y-s)}F(s,\mu)ds \tag{12}$$

or

$$\begin{bmatrix} I(y,\mu) \\ I(y,-\mu)\end{bmatrix} = e^{-E(\mu)(y-L)}\begin{bmatrix} I(L,\mu) \\ I(L,-\mu)\end{bmatrix} - \int_y^L e^{-E(\mu)(y-s)}F(s,\mu)ds, \tag{13}$$

where the exponential matrix $e^{-E(\mu)y}$ is equal to:

$$\frac{1}{2\gamma}\begin{bmatrix} \lambda(e^{-\frac{\gamma}{\mu}y} - e^{\frac{\gamma}{\mu}y}) + \gamma(e^{-\frac{\gamma}{\mu}y} + e^{\frac{\gamma}{\mu}y}) & \beta\left(e^{\frac{\gamma}{\mu}y} - e^{-\frac{\gamma}{\mu}y}\right) \\ -\beta\left(e^{\frac{\gamma}{\mu}y} - e^{-\frac{\gamma}{\mu}y}\right) & \lambda(e^{\frac{\gamma}{\mu}y} - e^{-\frac{\gamma}{\mu}y}) + \gamma(e^{-\frac{\gamma}{\mu}y} + e^{\frac{\gamma}{\mu}y}) \end{bmatrix}$$

with $\gamma = \sqrt{\lambda^2 - \beta^2}$.

Applying the boundary conditions (2), (3) in the second component of (12) and the first component of the (13) we obtain the intensity in the outer directions:

$$\begin{aligned} I(0,-\mu) = {} & \frac{2\gamma}{\lambda e^{\frac{\gamma}{\mu}L} - \lambda e^{-\frac{\gamma}{\mu}L} + \gamma e^{\frac{\gamma}{\mu}L} + \gamma e^{-\frac{\gamma}{\mu}L}} \\ & \cdot\left(B_L(-\mu) + \frac{1}{\mu}\int_0^L \frac{\beta\left(-e^{-\frac{\gamma}{\mu}(L-s)} + e^{\frac{\gamma}{\mu}(L-s)}\right)}{2\gamma}q(s,\mu)ds \right. \\ & \left. + \frac{1}{\mu}\int_0^L \frac{\lambda(e^{\frac{\gamma}{\mu}(L-s)} - e^{-\frac{\gamma}{\mu}(L-s)}) + \gamma(e^{-\frac{\gamma}{\mu}(L-s)} + e^{\frac{\gamma}{\mu}(L-s)})}{2\gamma}q(s,-\mu)ds\right) \\ & + \frac{\beta\left(e^{\frac{\gamma}{\mu}L} - e^{-\frac{\gamma}{\mu}L}\right)}{\lambda e^{\frac{\gamma}{\mu}L} - \lambda e^{-\frac{\gamma}{\mu}L} + \gamma e^{\frac{\gamma}{\mu}L} + \gamma e^{-\frac{\gamma}{\mu}L}}B_0(\mu) \end{aligned} \tag{14}$$

$$\begin{aligned} I(L,\mu) = {} & -B_L(-\mu)\frac{\beta\left(e^{-\frac{\gamma}{\mu}L} - e^{\frac{\gamma}{\mu}L}\right)}{-\lambda e^{-\frac{\gamma}{\mu}L} + \lambda e^{\frac{\gamma}{\mu}L} + \gamma e^{-\frac{\gamma}{\mu}L} + \gamma e^{\frac{\gamma}{\mu}L}} \\ & + \left(\frac{2\gamma}{-\lambda e^{-\frac{\gamma}{\mu}L} + \lambda e^{\frac{\gamma}{\mu}L} + \gamma e^{-\frac{\gamma}{\mu}L} + \gamma e^{\frac{\gamma}{\mu}L}}\right) \\ & \cdot\left(B_0(\mu) + \frac{1}{\mu}\int_0^L \frac{\lambda(e^{\frac{\gamma}{\mu}s} - e^{-\frac{\gamma}{\mu}s}) + \gamma(e^{\frac{\gamma}{\mu}s} + e^{-\frac{\gamma}{\mu}s})}{2\gamma}q(s,\mu)ds \right. \\ & \left. - \frac{1}{\mu}\int_0^L \frac{\beta\left(-e^{\frac{\gamma}{\mu}s} + e^{-\frac{\gamma}{\mu}s}\right)}{2\gamma}q(s,-\mu)ds\right) \end{aligned} \tag{15}$$

Then we substitute (14) and (15) into the system (12)–(13), we obtain the following integral formulation:

$$I(y,\mu) = L_g^{\mu}q(y,\mu) + L_b^{\mu}B. \tag{16}$$

where $B = (B_0, B_L)$.

Now we observe that the original problem (1)–(3) takes the form of (7)–(9) by assuming $q(y,\mu) = \sigma J(y,\mu) + Q(y)$ where J is defined by:

$$J(y,\mu) := \frac{1}{2}\int_{-1}^{1}\omega(\mu,\mu')I(y,\mu')d\mu'. \tag{17}$$

Taking into account Eq. (16), the solution of (1)–(3) satisfies

$$(19)I(y,\mu) = L_g^{\mu}[\sigma J(y,\mu) + Q(y)] + L_b^{\mu}B. \tag{18}$$

Substituting (19) in (17) we obtain:

$$\begin{aligned} J(y,\mu) &= \frac{\sigma}{2}\int_{-1}^{1}\left\{\left[\omega(\mu,\mu')L_g^{\mu'}J(y,\mu') + Q(y)\right] + \omega(\mu,\mu')L_b^{\mu'}B\right\}d\mu', \\ &:= L_g[\sigma J(y,\mu) + Q(y)] + L_b B, \end{aligned} \tag{19}$$

where the operators $L_g : C^0([0,L], L^\infty[-1,1]) \to C^0([0,L], L^\infty[-1,1])$ and $L_b : (L^\infty[0,1] \times L^\infty[-1,0]) \to C^0([0,L], L^\infty[-1,1])$ are given by