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On an analytical representation for the solution of the neutron point kinetics equation free of stiffness

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ABSTRACT

In this work, we report on an analytical representation for the solution of the neutron point kinetics equation, free of stiffness and assuming that reactivity is a continuous or sectionally continuous function of time. To this end, we cast the point kinetics equation in a first order linear differential equation. Next, we split the corresponding matrix into a diagonal matrix plus a matrix that contains the remaining terms. Expanding the neutron density and the delayed neutron precursor concentrations in a truncated series allows one to construct a recursive system in form of a first order matrix differential equation with source. The initialization of the recursion procedure is of diagonal form and has no source but satisfies the initial conditions. The remaining equations are subject to null initial conditions and include the time-dependent diagonal elements together with the off-diagonal elements as a source term. The solution is obtained in an analytical representation which may be evaluated for any time value, because it is free of stiffness. We present numerical simulations and comparisons against results from the literature for a constant, a step, a ramp, a quadratic, and a sine shaped reactivity function.

1. Introduction

In this work we focus on the neutron point kinetics equation, which models the variation of neutron population density responding to a prescribed reactivity function with time, considering six groups of neutron precursor contributions. This equation possesses, when considering constant reactivity (Li et al., 2010), a well-known analytical solution. However, when a time-dependent reactivity is considered, the solution for the neutron point kinetics equation can only be obtained by means of specific methods, known as A-stable methods, due to the fact that the relevant equation for the time-dependent reactivity features a stiffness problem (Petersen et al., 2011).

The literature presents several studies on the control of stiffness. As examples of the numerical approach of such techniques we cite: the method of finite elements (Zhiyuan, 1981), the Runge Kutta method (Sánchez, 1989), the quasi-static method (Koclas et al., 1997), the singular perturbation method (Bienski et al., 1978), and the finite difference method (Brown, 1967). It is noteworthy that all of the afore mentioned methods generate the time evolution of the neutron density and precursor concentrations by a finite interval time marching integration. Most of these methods work successfully in some specific applications, although they have disadvantages (Vigil, 1967) imposed by limitations on the maximum allowable time-step that still ensures computational stability. According to (Vigil, 1967) even the analytic continuation technique requires some modifications in its original form in order to achieve computational efficiency. Chao and Attardt (1985) developed the stiffness confinement method, and Basken and Lewis (1996) developed a solution for the neutron point kinetics equation based on a power series expansion, Aboander and Hamada (2004) made use of Padé approximants and exponential functions. In reference Nahla (2011) a generalized power series solution was developed, and in (Hamada, 2011) that solution was extended for several reactivity functions. One also finds some analytical methods in the literature; as examples of such methods we cite: (Bodmann et al., 2010; Chen et al., 2007; Petersen et al., 2011; Seliverstov and Shvedov, 1998 and Vilhena et al., 2008) all of which work with integral transforms, and (Ganapol, 2013) which made use of analytical approximations by Taylor series.

Our work is an attempt to eliminate the stiffness character in the solution of the neutron point kinetics equation by a decomposition-like method. To this end we solve the neutron point kinetics equation considering six groups of delayed neutron precursors and

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a time-dependent reactivity in form of a continuous or sectionally continuous function of time. The development of such a solution comprises the following steps: first, the neutron point kinetics equation is written as a matrix system of first order differential equations. Next, the system is decomposed into a separable time-independent part, and the matrix composed of the remaining elements of the original matrix also includes the time-dependent parts of the diagonal elements of the original matrix. The neutron density and the delayed neutron precursor concentrations are expanded in a truncated series and inserted in the matrix equation. From the obtained system one builds a recursive scheme that comprises the first order differential matrix equation with a source term. The separable system in diagonal matrix form is directly solved by incorporating the initial conditions, whereas from the first recursion step onwards, the source and the remaining terms are taken into account in order to determine the solution in analytical representation. The resulting recursive system incorporates in the source term the solution of the previous recursion and satisfies the null initial condition. Similar procedures were used in (Petersen et al., 2011).

It is noteworthy, that all (semi-) numerical schemes found in the literature, as for instance (Aboander and Hamada, 2004; Basken and Lewis, 1996; Chao and Attardt, 1985; Hamada, 2011; Nahla, 2011; Petersen et al., 2011 and Sánchez, 1989), need progressive time step techniques in order to evaluate the time dependence for the neutron density and precursor concentrations and are thus limited by the afore mentioned stiffness. The principal novelty of the present approach is that the obtained solution may be determined for any arbitrary time without any dependences of previous times; that is, the present procedure is non-progressive in time, and the only necessary specification to obtain numerical results are the details of integration by the Gauss–Legendre method, which does not impose any limits on time. Our findings are compared with those in the literature for several available methods. Additionally, convergence of our results is guaranteed by the application of a Lyapunov criterion discussed in (Boichenko et al., 2005) and also applied in (Petersen et al., 2011).

2. Matrix decomposition of neutron point kinetics

2.1. Matrix decomposition method

The neutron point kinetics equation with time-dependent reactivity free of truncation error is:

$$\begin{aligned}\frac{d}{dt}n(t) &= \frac{\rho(t)-\beta}{\Lambda}n(t)+\sum_{i=1}^{6}\lambda_i C_i(t),\ n(0)=n_0,\\ \frac{d}{dt}C_i(t) &= \frac{\beta_i}{\Lambda}n(t)-\lambda_i C_i(t),\ C_i(0)=\frac{\beta_i n_0}{\Lambda\lambda_i},\end{aligned} \tag{1}$$

for $i\in[1,6]$. Here $n(t)$ denotes the neutron density, $\rho(t)$ represents the time-dependent reactivity, β_i represents the fraction of delayed neutrons produced in a specific group of energy i, β the fraction of the delayed neutrons, λ_i are the decay constants of the precursors, Λ represents the neutron mean generation time, and $C_i(t)$ is the density of the i-th delayed neutron precursor group.

The solution, which will be determined in closed form using the above mentioned recursive scheme, will be used with finite recursive depth R for numerical applications. The truncation will be determined such that the solution shows exponential convergence in the time interval of interest. To this end we write the neutron and the precursors densities in terms of the solution by a recursion initialization ($j=0$) and the respective correction terms ($j>0$).

$$n(t)=\sum_{j=0}^{R}n_j(t),\ C_i(t)=\sum_{j=0}^{R}C_{ij}(t),\quad \forall R\in\mathbb{N}. \tag{2}$$

In addition, the reactivity will be expressed as $\rho(t)=\rho_0+\rho_1(t)$, with ρ_0 being a constant value, and, $\rho_1(t)$ a time-dependent function. Thus, Eq. (1) leads to

$$\begin{aligned}\left(\frac{d}{dt}-\frac{\rho_0-\beta}{\Lambda}\right)\sum_{j=0}^{R}n_j(t) &= \frac{\rho_1}{\Lambda}\sum_{j=0}^{R}n_j(t)+\sum_{i=1}^{6}\lambda_i\sum_{j=0}^{R}C_{ij}(t),\\ \left(\frac{d}{dt}+\lambda_i\right)\sum_{j=0}^{R}C_{ij}(t) &= \frac{\beta_i}{\Lambda}\sum_{j=0}^{R}n_j(t),\end{aligned} \tag{3}$$

which may be cast in matrix form.

$$\frac{d\mathbf{Y}(t)}{dt}-\Omega\mathbf{Y}(t)=\Xi(t)\mathbf{Y}(t) \tag{4}$$

Formally, Eq. (4) may be solved by Laplace transform.

$$\mathbf{Y}(t)=\exp(\Omega t)\mathbf{Y}(0)+\int_0^t\exp(\Omega\tau)\Xi(t-\tau)\mathbf{Y}(t-\tau)\,d\tau \tag{5}$$

Note that the integral contains the time-dependent reactivity term together with the decay constants that gave origin to the stiffness character of the problem. In general, analytical solutions of the integral are possible only for specific time dependencies of reactivity, so that one has to resort to other techniques in order to solve the integral for any arbitrary time dependence of $\rho_1(t)$. To this end we decompose the integral Eq. (5) into a set of equations following a reasoning similar to that which led Adomian to propose his decomposition method; that is, the original problem is cast in a recursion scheme, where the initialization can be solved directly and the remaining equations are solved by the Gauss–Legendre integration method. The recursion depth is determined from the desired numerical accuracy.

For convenience we introduced the matrices:

$$\mathbf{Y}_j(t)=\left(n_j(t),C_{1j}(t),C_{2j}(t),C_{3j}(t),C_{4j}(t),C_{5j}(t),C_{6j}(t)\right)^T, \tag{6}$$

$$\mathbf{\Omega}=\mathrm{diag}\left(\frac{\rho_0-\beta}{\Lambda},-\lambda_1,-\lambda_2,-\lambda_3,-\lambda_4,-\lambda_5,-\lambda_6\right), \tag{7}$$

and

$$\mathbf{\Xi}=\begin{bmatrix}\frac{\rho_1(t)}{\Lambda} & \{\lambda_i\}\\ \left\{\frac{\beta_i}{\Lambda}\right\}^T & 0\end{bmatrix}, \tag{8}$$

such that the recursion initialization is

$$\frac{d\mathbf{Y}_0(t)}{dt}-\Omega\mathbf{Y}_0(t)=0, \tag{9}$$

and the closed form solution is given by the recursive scheme

$$\frac{d\mathbf{Y}_i(t)}{dt}-\Omega\mathbf{Y}_i(t)=\Xi(t)\mathbf{Y}_{i-1}(t). \tag{10}$$

In the following we indicate the operational details of the afore presented scheme:

- Matrix Eq. (4) is the original problem (1) cast in matrix form with the initial condition $\mathbf{Y}(0)$ determined from the specified condition $n_0=n(0)=1\ \mathrm{cm}^{-3}$ and the relation $C_i(0)=\frac{\beta_i}{\Lambda\lambda_i}n_0$ of Eqs. (1).
- The solution after R recursion steps is $\mathbf{Y}^{(R)}(t)=\sum_{i=0}^{R}\mathbf{Y}_i(t)$ and combines the decomposition series of Eq. (2).
- The original problem is decomposed into an in principle infinite number of equations, that set up a recursive scheme:
  - The initialization of the scheme is based on a separation of the original problem (4) into a part with known solution, which in the present case is determined by the time independent part with diagonal matrix Ω shown in (9) and the remaining terms commented in the next item. The solution of (9) is the first term on the right hand side in Eq. (5).