Annals of Nuclear Energy 70 (2014) 1–10

Contents lists available at ScienceDirect

Annals of Nuclear Energy

journal homepage: www.elsevier.com/locate/anucene

On the numerical solution of the neutron fractional diffusion equation

Nader Maleki Moghaddam [a], Hossein Afarideh [a,*], Gilberto Espinosa-Paredes [b]

[a] Amir Kabir University of Technology (Tehran Polytechnic), Department of Energy Engineering & Physics, Hafez Av., P.O. Box 15875-4413, Tehran, Iran
[b] Área deIngeniería en Recursos Energetic, Universidad Autónoma Metropolitana-Iztapalapa, Av. San Rafael Atlixco 186 Col. Vicentina, México, DF 09340, Mexico

ARTICLE INFO

Article history:
Received 2 October 2013
Received in revised form 17 January 2014
Accepted 21 January 2014
Available online 20 March 2014

Keywords:
Neutron fractional diffusion equation
Space history
Non-local gradient
Fractional calculus

ABSTRACT

In order to core calculation in the nuclear reactors there is a new version of neutron diffusion equation which is established on the fractional partial derivatives, named Neutron Fractional Diffusion Equation (NFDE). In the NFDE model, neutron flux in each zone depends directly on the all previous zones (not only on the nearest neighbors). Under this circumstance, it can be said that the NFDE has the space history. We have developed a one-dimension code, NFDE-1D, which can simulate the reactor core using arbitrary exponent of differential operators. In this work a numerical solution of the NFDE is presented using shifted Grünwald-Letnikov definition of fractional derivative in finite differences frame. The model is validated with some numerical experiments where different orders of fractional derivative are considered (e.g. 0.999, 0.98, 0.96, and 0.94). The results show that the effective multiplication factor (K_{eff}) depends strongly on the order of fractional derivative.

1. Introduction

Fractional calculus is three centuries old, as much as conventional calculus, but not very popular amongst the science and engineering community. The beauty of this subject is that fractional derivatives (and integrals) are not a local (or point) property (or quantity) (Das and Basudeb, 2007). Therefore, it considers the history and non-local distributed effects. Over a few decades this subject has been of interest by scientists and engineers, due that the fractional calculus may reflect the nature of the phenomena. In the nuclear reactor there is a new version of the neutron diffusion equation, which is established on the fractional derivatives. In a recent work (Espinosa-paredes et al., 2013), a Fractional-Space law of neutron diffusion equation is introduced. In the fractional diffusion equation the *Non-local* gradient of Fick's law is used ($\mathbf{J} = -D_\alpha \nabla^\alpha \phi$), where the fractional derivative operator ∇^α can be defined in the Riemann–Liouville, Caputo and Grünwald-Letnikov sense (Das, 2012; Oldham and Spanier, 1974) and α is the order of derivative. In the limit $\alpha \to 1$, the Ficḱs law is recovered. Fractional calculus deals with the study of the fractional order of the integral and derivative operators over real or complex domains, and its application in various fields of science and engineering is reported in literatures (e.g., Balakrishnan, 1985; Oldham and Spanier, 1974; Miller and Ross, 1993; Samko et al., 1993; Podlubny, 1999; Hilfer, 2000; Kilbas et al., 2006; Magin, 2006; Das and Biswas, 2005). Until now, there is no reported numerical solution of the NFDE so the full numerical scheme of neutron fractional diffusion equation is described based on the shifted Grünwald-Letnikov expression. The numerical solution of the NFDE in two energy groups in a slab geometry is presented. Finally the results of the NFDE with different orders of fractional derivatives (α) are compared with results obtained by the neutron classical diffusion equation.

2. Fractional calculus

2.1. History

Scientists and engineers meet with differential operators such as $\frac{\partial}{\partial x}, \frac{\partial^2}{\partial x^2}$, sand so on, however few of them consider about whether if it is necessary for the order of differentiation to be an integer. Why not be a rational, fractional, irrational, or even a complex number? At the very beginning of integral and differential calculus, in a letter to LHopital in 1695, Leibniz himself raised the question: Can the meaning of derivatives with integer order be generalized to derivatives with non-integer orders? LHopital was somewhat curious about that question and replied by another question to Leibniz: What if the order will be 1/2? Leibniz in a letter dated September 30, 1695 replied: It will lead to a paradox, from which, one day, useful consequences will be drawn. The question raised by Leibniz for a non-integer-order derivative was an ongoing topic for more than 300 years, and now it is known as *fractional calculus*, a generalization of ordinary differentiation and integration to arbitrary (non-integer) order (Monje et al., 2010; Das, 2012).

* Corresponding author. Tel.: +98 21 64545230; fax: +98 21 66495519.
E-mail address: Hafarideh@aut.ac.ir (H. Afarideh).

http://dx.doi.org/10.1016/j.anucene.2014.01.044
0306-4549/

2.2. Famous relation of fractional integral/differential

In order to achieve a fractional order of differentiation/integration operator we should effort to interpolate the operators between two integer order operations, so in the limit sense when the order of operator approaches to integer, the differentiation/integration operations inclines to classical integer operations. There are three famous relations of fractional integral/differential, in the next subsections these are briefly described.

2.2.1. Riemann–Liouville's definition of fractional integral

Consider an anti-derivative or primitive of the function $\phi(x)$, $D^{-1}\phi(x)$, then:

$$D^{-1}\phi(x) = \int_0^x \phi(\tau)d\tau \tag{1}$$

The calculation of the second order integration can be simplified by interchanging the integration order (Fig. 1).

$$\begin{aligned} D^{-2}\phi(x) &= \int_0^x \int_0^{\tau_2} \phi(\tau_1)d\tau_1 d\tau_2 = \int_0^x \int_{\tau_1}^x \phi(\tau_1)d\tau_2 d\tau_1 \\ &= \int_0^x \phi(\tau_1)\int_{\tau_1}^x d\tau_2 d\tau_1 = \int_0^x \phi(\tau)(x-\tau)d\tau \end{aligned} \tag{2}$$

This method can be applied repeatedly then:

$$D^{-3}\phi(x) = \frac{1}{2}\int_0^x (x-\tau)^2\phi(\tau)d\tau \tag{3}$$

For nth order of integration:

$$D^{-n}\phi(x) = \frac{1}{(n-1)!}\int_0^x (x-\tau)^{n-1}\phi(\tau)d\tau \tag{4}$$

The last equation, in which we can see that an iterated integral may be expressed as a weighted single integral with a very simple weighting function, is known as the Cauchy's formula for iterated or repeated integral. If we generalize Eq. (4) for the case of $\alpha \in R^+$, we obtain

$$D^{-\alpha}\phi(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-\tau)^{\alpha-1}\phi(\tau)d\tau \tag{5}$$

Which is corresponds to the Riemann–Liouville's definition for the fractional order integral of order $\alpha \in R^+$.

2.2.2. Riemann–Liouville's definition of fractional differential

Riemann–Liouville definited the fractional differential as follows:

$$D^{\alpha}\phi(x) = \frac{d^m}{dx^m}\left[\frac{1}{\Gamma(m-\alpha)}\int_0^x \frac{\phi(\tau)}{(x-\tau)^{\alpha+1-m}}d\tau\right] \\ (m-1) \leqslant \alpha < m \tag{6}$$

where m is the integer and α is a positive real number. Eq. (6) is known as the Left hand Riemann–Liouville method (RL-Left Hand). The formulation of this definition is described below:

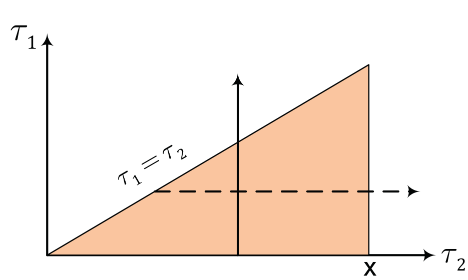

Fig. 1. Schematic of interchanging the integration order.

Select an integer m greater than fractional number α then

(i) Integrate the function $(m-\alpha)$ folds.
(ii) Differentiate the above result by m.

To illustrate steps of fractional differentiation in the Left hand definition refer to Fig. 2 which shows the fractional differentiation of 2.3 times in the Left Hand Definition (LHD) of RL.

2.2.3. Caputo's definition of fractional differential

Caputo formulated the fractional differentials as follow:

$$\begin{aligned} D^{\alpha}\phi(x) &= \left[\frac{1}{\Gamma(m-\alpha)}\int_0^x \frac{\frac{d\phi(\tau)^m}{d\tau^m}}{(x-\tau)^{\alpha+1-m}}d\tau\right] \\ &= \frac{1}{\Gamma(m-\alpha)}\int_0^x \frac{\phi^{(m)}(\tau)}{(x-\tau)^{\alpha+1-m}}d\tau \\ &(m-1) \leqslant \alpha < m \end{aligned} \tag{7}$$

where m is the integer and α is a positive real number. This formulation is exactly opposite to LHD. So it is known as the right hand definition (RHD), for RHD of fractional differentials first select an integer m greater than fractional number then:

(i) Differentiate the function m times.
(ii) Integrate the above result $(m-\alpha)$ fold by RL integration method. To illustrate steps of the fractional differentiation in the right hand definition refer to Fig. 3 which shows the fractional differentiation of 2.3 times in RHD.

2.2.4. Grünwald-Letnikov's definition of fractional differential

The process of Grünwald-Letnikov definition is described below:

$$\phi'(x) = \lim_{h\to 0}\frac{\phi(x+h)-\phi(x)}{h} \tag{8}$$

$$\begin{aligned} \phi''(x) &= \lim_{h\to 0}\frac{\phi'(x+h)-\phi'(x)}{h} \\ &= \lim_{h_1\to 0}\frac{\lim_{h_2\to 0}\frac{\phi(x+h_1+h_2)-\phi(x+h_1)}{h_2} - \lim_{h_2\to 0}\frac{\phi(x+h_2)-\phi(x)}{h_2}}{h_1} \end{aligned} \tag{9}$$

If $h = h1 = h2$ then:

$$\phi''(x) = \lim_{h\to 0}\frac{\phi(x+2h)-2\phi(x+h)+\phi(x)}{h^2} \tag{10}$$

Continuing for "n" times we will have:

$$D^n\phi(x) = \lim_{h\to 0}\frac{1}{h^n}\sum_{m=0}^{n}(-1)^m\binom{n}{m}\phi(x-mh) \tag{11}$$

where $\binom{n}{m} = \frac{n!}{m!(n-m)!}$

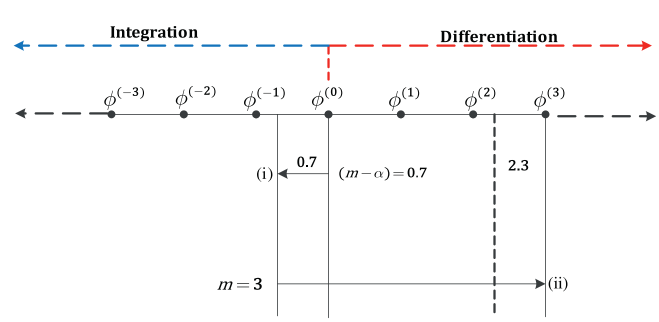

Fig. 2. Fractional differentiation of 2.3 times in LHD.