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Two-dimensional Haar wavelet Collocation Method for the solution of Stationary Neutron Transport Equation in a homogeneous isotropic medium

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ABSTRACT

This paper emphasizes on finding the solution for a stationary transport equation using the technique of Haar wavelet Collocation Method (HWCM). Haar wavelet Collocation Method is efficient and powerful in solving wide class of linear and nonlinear differential equations. Recently Haar wavelet transform has gained the reputation of being a very effective tool for many practical applications. This paper intends to provide the great utility of Haar wavelets to nuclear science problem. In the present paper, two-dimensional Haar wavelets are applied for solution of the stationary Neutron Transport Equation in homogeneous isotropic medium. The proposed method is mathematically very simple, easy and fast. To demonstrate about the efficiency of the method, one test problem is discussed. It can be observed from the computational simulation that the numerical approximate solution is much closer to the exact solution.

1. Introduction

Wavelet Analysis is a new branch of mathematics and widely applied in signal analysis, image processing and numerical analysis, etc. (Chang and Piau, 2008; Chen et al., 2010). Among the different wavelet families mathematically most simple are the Haar wavelets (Lepik, 2009). In 1910, Alfred Haar introduced the notion of wavelets. Haar wavelets have the properties of orthogonal and normalization having close support and the simple expression (Li and Zhao, 2010; Saha Ray, 2012). Due to its simplicity, the Haar wavelets are very efficient and effective tool for solving both differential and integral equations.

Integro-differential equations (IDEs) have many applications in different fields of mechanical, nuclear engineering, chemistry, astronomy, biology, economics, potential theory and electrostatics. An exact solution of this integro-differential equation was found only in the particular cases. In many cases analytical solution of IDEs is unwieldy task; therefore our aim is focused on exploring accurate and efficient numerical method (Islam et al., 2013).

The motivation of the present paper is to solve a typical problem of the mathematical–physics: the solving a particle transport equation that has numerous applications in physics and astrophysics (Martin, 2011). In the reactor, the neutrons are generated at the fission of the nucleus and they are named as rapid neutrons with an average speed equals to 2×10^7 m/s. In the stationary state of the reactor, the particles have the tendency to move from a region with a large density to another with a small density which yields a uniform density. The main emphasis in the reactor theory is to find the neutrons distribution in the reactor and hence its density which is the solution of an integro-differential equation known as neutron transport equation.

In this study, we consider a linear form of Boltzmann equation with a source function of the form $f(x,\eta) = A(\eta)\cos\pi\eta + B(\eta)$ - $\sin\pi\eta$. To obtain the solution of this stationary transport equation, we have applied Haar wavelet transform method. A numerical example will lead us to the conclusion on the advantage of this method.

2. Formulation of neutron transport equation model

Let us consider the integral–differential equation for the stationary case of transport theory (Martin, 2011)

$$\eta \frac{\partial \phi(x,\eta)}{\partial x} + \phi(x,\eta) = \frac{1}{2}\int_{-1}^{1} \phi(x,\eta')d\eta' + f(x,\eta), \tag{2.1}$$

$$\forall (x,\eta) \in D_1 \times D_2 = [0,1] \times [-1,1], D_2 = D_2' \cup D_2'' = [-1,0] \cup [0,1]$$

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with the following boundary conditions:

$$\phi(0,\eta) = 0 \quad \text{if } \eta > 0 \quad \text{and } \phi(1,\eta) = 0 \quad \text{if } \eta < 0 \tag{2.2}$$

where $\phi(x, \eta)$ is the neutron density which migrate in a direction which makes an angle α with the x-axis and $\eta = \cos\alpha$; $f(x, \eta)$ is a given radioactive source function.

Now, we split the Eq. (2.1) into two equations using the following notations

$$\phi^+ = \phi(x,\eta) \quad \text{if } \eta > 0 \quad \text{and } \phi^- = \phi(x,\eta) \quad \text{if } \eta < 0 \tag{2.3}$$

By denoting, $\eta' = -\eta$, we can obtain

$$\int_{-1}^{0} \phi(x,\eta')d\eta' = \int_0^1 \phi(x,-\eta)d\eta = \int_0^1 \phi^- d\eta$$

In view of Eq. (2.3), Eq. (2.1) can be written as

$$\eta\frac{\partial\phi^+}{\partial x} + \phi^+ = \frac{1}{2}\int_0^1 (\phi^+ + \phi^-)d\eta' + f^+ \quad \text{for } \eta > 0 \tag{2.4}$$

$$-\eta\frac{\partial\phi^-}{\partial x} + \phi^- = \frac{1}{2}\int_0^1 (\phi^+ + \phi^-)d\eta' + f^- \quad \text{for } \eta < 0 \tag{2.5}$$

with the boundary conditions $\phi^+(0, \eta) = 0$, $\phi^-(1, \eta) = 0$.

Adding and subtracting the Eqs. (2.4) and (2.5) and then introducing the following notations, we obtain

$$u = \frac{1}{2}(\phi^+ + \phi^-),\ v = \frac{1}{2}(\phi^+ - \phi^-),\ g = \frac{1}{2}(f^+ + f^-) \quad \text{and}$$
$$r = \frac{1}{2}(f^+ - f^-) \tag{2.6}$$

We also obtain the following system

$$\eta\frac{\partial v}{\partial x} + u = \int_0^1 u d\eta + g, \tag{2.7}$$

$$\eta\frac{\partial u}{\partial x} + v = r \tag{2.8}$$

along with the following boundary conditions

$$u + v = 0 \quad \text{for } x = 0,$$

$$u - v = 0 \quad \text{for } x = 1. \tag{2.9}$$

Eliminating the value of v from Eqs. (2.7) and (2.8), we rewrite the problem (2.7)–(2.9) in the following form

$$-\eta^2\frac{\partial^2 u}{\partial x^2} + u = \int_0^1 u d\eta + g - \eta\frac{\partial r}{\partial x}$$

$$\left(u - \eta\frac{\partial u}{\partial x}\right)\Big|_{x=0} = -r(0,\eta), \tag{2.10}$$

$$\left(u + \eta\frac{\partial u}{\partial x}\right)\Big|_{x=1} = r(1,\eta), \tag{2.11}$$

where $\eta \in [0, 1]$.

3. Haar wavelets

Haar functions have been used from 1910 when they were introduced by the Hungarian mathematician Alfred Haar. Haar wavelets are the simplest wavelets among various types of wavelets. They are step functions over the real line can take only three values 0, 1 and −1. The method has been used for being its simpler, fast and computationally attractive feature. The Haar functions are the family of switched rectangular waveforms where amplitudes can differ from one function to another function. Usually the Haar wavelets are defined for the interval $t \in [0, 1)$ but in general case $t \in [A, B]$, we divide the interval $[A, B]$ into m equal subintervals; each of width $\Delta t = (B - A)/m$. In this case, the orthogonal set of Haar functions are defined in the interval $[A, B]$ by (Saha Ray, 2012; Saha Ray and Patra, 2013)

$$h_0(t) = \begin{cases} 1 & t \in [A,B], \\ 0 & \text{elsewhere}, \end{cases}$$

$$\text{and} \quad h_i(t) = \begin{cases} 1, & \zeta_1(i) \le t < \zeta_2(i) \\ -1, & \zeta_2(i) \le t < \zeta_3(i) \\ 0, & \text{otherwise} \end{cases} \tag{3.1}$$

$$\text{where} \quad \zeta_1(i) = A + \left(\frac{k-1}{2^j}\right)(B-A) = A + \left(\frac{k-1}{2^j}\right)m\Delta t,$$

$$\zeta_2(i) = A + \left(\frac{k-(1/2)}{2^j}\right)(B-A) = A + \left(\frac{k-(1/2)}{2^j}\right)m\Delta t,$$

$$\zeta_3(i) = A + \left(\frac{k}{2^j}\right)(B-A) = A + \left(\frac{k}{2^j}\right)m\Delta t,$$

$i = 1, 2, \ldots, m$, $m = 2^J$ and J is, a positive integer, called the maximum level of resolution. Here, j and k represent the integer decomposition of the index i. i.e. $i = k + 2^j - 1$, $0 \leqslant j < i$ and $1 \leqslant k < 2^j + 1$.

4. Function approximation

Any function $y(t) \in L^2([0, 1))$ can be expanded in Haar series as

$$y(t) = c_0h_0(t) + c_1h_1(t) + c_2h_2(t) + \ldots,$$

$$\text{where} \quad c_j = \int_0^1 y(t)h_j(t)dt. \tag{4.1}$$

If $y(t)$ is approximated as piecewise constant in each subinterval, the sum in Eq. (4.1) may be terminated after m terms and consequently we can write discrete version in the matrix form as

$$\boldsymbol{Y} \approx \sum_{i=0}^{m-1} c_i h_i(t_l) = \boldsymbol{C}_{\boldsymbol{m}}^{\boldsymbol{T}} H_m, \tag{4.2}$$

$$\text{for collocation points } t_l = A + (l - 0.5)\Delta t,\ l = 1, 2, \ldots, m \tag{4.3}$$

where $\boldsymbol{Y}$ and $\boldsymbol{C}_{\boldsymbol{m}}^{\boldsymbol{T}}$ are the m-dimensional row vectors.

Here H is the Haar wavelet matrix of order m defined by $H = [\boldsymbol{h_0}, \boldsymbol{h_1}, \ldots, \boldsymbol{h_{m-1}}]^T$, i.e.

$$H = \begin{bmatrix} \boldsymbol{h_0} \\ \boldsymbol{h_1} \\ \cdots \\ \boldsymbol{h_{m-1}} \end{bmatrix} = \begin{bmatrix} h_{0,0} & h_{0,1} & \cdots & h_{0,m-1} \\ h_{1,0} & h_{1,1} & \cdots & h_{1,m-1} \\ \cdots & & & \\ h_{m-1,0} & h_{m-1,1} & \cdots & h_{m-1,m-1} \end{bmatrix}, \tag{4.4}$$

where $\boldsymbol{h_0}, \boldsymbol{h_1}, \ldots, \boldsymbol{h_{m-1}}$ are the discrete form of the Haar wavelet bases.

5. Operational matrix of the general order integration

The integration of the $H_m(t) = [h_0(t), h_1(t), \ldots, h_{m-1}(t)]^T$ can be approximated by Chen and Hsiao (1997)

$$\int_0^t H_m(\tau)d\tau \cong QH_m(t), \tag{5.1}$$

where Q is called the Haar wavelet operational matrix of integration which is a square matrix of m-dimension. To derive the Haar wavelet operational matrix of the general order of integration, we recall the fractional integral of order $\alpha(>0)$ which is defined by Podlubny (1999)