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Exact analytical solution of time-independent neutron transport equation, and its applications to systems with a point source

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ABSTRACT

An exact analytical solution of the time-independent monoenergetic neutron transport equation is obtained in this paper. The solution is applied to systems with a point source. Systematic analysis of the solution of the time-independent neutron transport equation, and its applications represent the primary goal of this paper. To the best of the author's knowledge, certain key results on the scalar neutron flux as well as their derivations are new. As an application of these results, a scalar neutron flux for a purely absorbing medium with a spherically piecewise constant cross section and an isotropic point neutron source off the origin as well as that for a cylindrically piecewise constant cross section with a point neutron source off the origin are obtained. Both of these results are believed to be new.

1. Introduction

This paper describes an exact solution of the time-independent one-speed neutron transport equation, and its applications to systems with a point source. The primary purpose of this paper is to systematically treat the solution of the time-independent neutron transport equation, and its applications. Even though the solution of the time-independent neutron transport equation has been widely discussed in the literature (e.g., Davison and Sykes, 1957; Case and Zweifel, 1967; Duderstadt and Martin, 1979; Ganapol, 2008), some of the results in this paper do not seem to have been published. To the best of the author's knowledge, certain key results on the scalar neutron flux as well as their derivations are new. Furthermore, some of the applications of these results are also new, in particular, the results for spherically and cylindrically piecewise constant macroscopic cross sections with an isotropic point source off the origin in a purely absorbing medium.

Developing analytical results and obtaining solutions for particular material and geometrical systems are important in regard to the verification of codes, since they will provide benchmark solutions for the codes. In the present paper a scalar neutron flux has been obtained for a purely absorbing medium with a variable macroscopic cross section (i.e., spatial variable dependent macroscopic cross section), when the neutron source density is given. This solution can be used to obtain special solutions for a variety of material and geometrical systems. Particular systems explored in this paper are a material system with a spherically piecewise macroscopic cross section and an isotropic point neutron source as well as a material system with a cylindrically piecewise macroscopic cross section with an isotropic point neutron source. However, the general solution obtained in the present paper can be used to other material and geometrical systems with point neutron sources as well as non-point (i.e., distributed) neutron sources.

2. Problem statement

Let us consider the time-independent one-speed neutron transport equation, which is given by

$$\boldsymbol{\Omega}\cdot\nabla\psi(\boldsymbol{r},\boldsymbol{\Omega})+\Sigma(\boldsymbol{r})\psi(\boldsymbol{r},\boldsymbol{\Omega})=Q(\boldsymbol{r},\boldsymbol{\Omega}) \quad (1)$$

where $\psi(\boldsymbol{r},\boldsymbol{\Omega})$ is the angular neutron flux, $\boldsymbol{r}$ is a position vector, $\boldsymbol{\Omega}$ is a unit vector in the direction of the neutron motion, $\Sigma(\boldsymbol{r})$ is the total macroscopic cross section, and $Q(\boldsymbol{r},\boldsymbol{\Omega})$ is the sum of the source and the scattering terms. Our goal is to derive a general solution for Eq. (1), converting that solution to an expression for the scalar neutron flux in terms of volume integral, and to use the result to obtain specific analytical results for a variety of material systems with a point source.

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3. Mathematical formulation

Let us introduce a new variable R, which is geometrically represented in Fig. 1.

Using this new and additional variable, R, Eq. (1) can be written as an ordinary differential equation in terms of R as follows.

$$-\frac{d}{dR}\psi(\boldsymbol{r}-R\boldsymbol{\Omega},\boldsymbol{\Omega})+\Sigma(\boldsymbol{r}-R\boldsymbol{\Omega})\psi(\boldsymbol{r}-R\boldsymbol{\Omega},\boldsymbol{\Omega})=Q(\boldsymbol{r}-R\boldsymbol{\Omega},\boldsymbol{\Omega}) \tag{2}$$

with

$$\psi(\boldsymbol{r},\boldsymbol{\Omega})=\psi(\boldsymbol{r}-R\boldsymbol{\Omega},\boldsymbol{\Omega})|_{R=0} \tag{3}$$

Eq. (2) can be easily integrated in terms of R, and after some manipulations involving the applications of the free surface boundary condition or the boundary condition at infinity following Davison and Sykes (1957), we obtain

$$\begin{aligned}\psi(\boldsymbol{r},\boldsymbol{\Omega})&=\psi(\boldsymbol{r}-R\boldsymbol{\Omega},\boldsymbol{\Omega})|_{R=0}\\&=\int_0^\infty dR' e^{-\int_0^{R'}\Sigma(\boldsymbol{r}-u\boldsymbol{\Omega})du}Q(\boldsymbol{r}-R'\boldsymbol{\Omega},\boldsymbol{\Omega})\end{aligned} \tag{4}$$

where R' is a dummy variable, and it is noted that the substitution of $R = 0$ has already been performed. It should be pointed out, however, that the above Eq. (4) is more general than the one derived in Davison and Sykes (1957), since Davison obtained the result only for an isotropic source with a constant macroscopic cross section. It can be shown that the above result is equivalent to a solution given in Duderstadt and Martin (1979), if Q is replaced by the source term S. Duderstadt and Martin (1979), however, did not show the detail of the derivation. Eq. (4) is a fundamental result for Eq. (1), and from which an integral equation for $\psi(r, \Omega)$ can be derived in general. Here instead of $\psi(\boldsymbol{r}, \boldsymbol{\Omega})$, we will focus on the scalar neutron flux, which is given by

$$\Phi(\boldsymbol{r})=\int_{4\pi}d\Omega\psi(\boldsymbol{r},\boldsymbol{\Omega}) \tag{5}$$

where the differential steradian $d\Omega$ is defined at the location Ω relative to the neutron, and they are defined in terms of local spherical coordinates located at the neutron as

$$\begin{aligned}d\Omega&=\sin\theta d\theta d\phi\\\boldsymbol{\Omega}&=(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta)\end{aligned} \tag{6}$$

Accordingly, the integral over the steradian Ω is defined as

$$\int_{4\pi}d\Omega=\int_0^\pi d\theta\int_0^{2\pi}d\phi\sin\theta \tag{7}$$

Substituting (4) into (5), we obtain

$$\Phi(\boldsymbol{r})=\int_0^\infty dR'\int_{4\pi}d\Omega e^{-\int_0^{R'}\Sigma(\boldsymbol{r}-u\boldsymbol{\Omega})du}Q(\boldsymbol{r}-R'\boldsymbol{\Omega},\boldsymbol{\Omega}) \tag{8}$$

Let us introduce a new variable

$$\boldsymbol{r}'=\boldsymbol{r}-R'\boldsymbol{\Omega} \tag{9}$$

From (9), we have

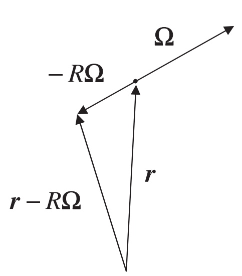

Fig. 1. Introduction of a new variable R.

$$\begin{aligned}R'&=|\boldsymbol{r}-\boldsymbol{r}'|\\\Omega&=\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|}\end{aligned} \tag{10}$$

It should be also noted that

$$\int_0^\infty dR'\int_{4\pi}d\Omega=\iiint_V\frac{d\boldsymbol{r}'}{(R')^2}=\iiint_V\frac{d\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^2} \tag{11}$$

where V is the entire three dimensional Euclidean space. Using Eqs. (9)–(11) in (8), we finally obtain

$$\Phi(\boldsymbol{r})=\int_v\frac{d\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^2}e^{-|\boldsymbol{r}-\boldsymbol{r}'|\int_0^1 dv\Sigma((1-v)\boldsymbol{r}+v\boldsymbol{r}')}Q\left(\boldsymbol{r}',\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|}\right) \tag{12}$$

where the volume integral is performed in the entire three dimensional Euclidean space. This is a fundamental result for the scalar neutron flux, which can be used to derive an integral equation for the scalar neutron flux $\Phi(\boldsymbol{r})$. In the case of purely absorbing media, $Q(\boldsymbol{r}', \boldsymbol{\Omega})$ can be replaced with a source density $S(\boldsymbol{r}', \boldsymbol{\Omega})$. In that case, instead of an integral equation, we obtain a closed-form solution for $\Phi(\boldsymbol{r})$ as

$$\Phi(\boldsymbol{r})=\int_v\frac{d\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^2}e^{-|\boldsymbol{r}-\boldsymbol{r}'|\int_0^1 dv\Sigma((1-v)\mathrm{bir}+v\boldsymbol{r}')}S\left(\boldsymbol{r}',\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|}\right) \tag{13}$$

If the macroscopic cross section is constant, Eq. (13) reduces to

$$\Phi(\boldsymbol{r})=\int_v\frac{d\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^2}e^{-\Sigma|\boldsymbol{r}-\boldsymbol{r}'|}S\left(\boldsymbol{r}',\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|}\right) \tag{14}$$

The above two equations (13) and (14) are used in the following sections to obtain specific results for the scalar neutron flux. The above two results (12) and (13), and their derivations seem to have never been published in the literature.

4. Point source with a variable macroscopic cross section

The source density for an isotropic point source with the source strength S_0 neutrons per second located at $\boldsymbol{r} = \boldsymbol{a}$ can be expressed as

$$S(\boldsymbol{r},\boldsymbol{\Omega})=\frac{S_0}{4\pi}\delta(\boldsymbol{r}-\boldsymbol{a})=\frac{S_0}{4\pi}\delta(r-a_x)\delta(r-a_y)\delta(r-a_z) \tag{15}$$

where $\delta(\boldsymbol{r} - \boldsymbol{a})$ is Dirac delta function in 3D. Substituting (15) into (13), we obtain

$$\Phi(r)=\frac{S_0}{4\pi|\boldsymbol{r}-\boldsymbol{a}|^2}e^{-|\boldsymbol{r}-\boldsymbol{a}|\int_0^1 dv\Sigma((1-v)\boldsymbol{r}+v\boldsymbol{a})} \tag{16}$$

This is a general result for the scalar neutron flux with an isotropic point source located at $\boldsymbol{r} = \boldsymbol{a}$ with the source strength S_0 in a material with a variable macroscopic cross section. This result is used in the following subsections to obtain particular results.

4.1. Spherically piecewise constant macroscopic cross section with an isotropic point source at the origin

Let us consider a piecewise constant macroscopic cross section given by

$$\Sigma(r)=\begin{cases}\Sigma_1 & 0<|\boldsymbol{r}|<r_1\\\Sigma_2 & r_1<|\boldsymbol{r}|<r_2\\\vdots & \\\Sigma_n & r_{n-1}<|\boldsymbol{r}|<r_n\\\Sigma_{n+1} & r_n<|\boldsymbol{r}|<\infty\end{cases} \tag{17}$$

where r_k is the outer radius of the kth spherical shell. Substituting $\boldsymbol{a} = \boldsymbol{0}$ and (17) into (16), we obtain