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A comparison of regularization operators for noisy gamma-ray tomographic reconstruction

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A B S T R A C T

The severe effects of several types of noise present in tomographic image reconstruction are well known. Some of these are a consequence of the ill-posedness of this inverse problem. In this work, we investigate the impact of a Tikhonov regularization on the solution of a gamma-ray tomography reconstruction by means of a least squares numerical method. The theoretical methodology is considered in a broad sense as a Tikhonov regularization, but also includes the Morozov concept used specifically for the delta parameter control. The reconstruction quality shows effective improvement when this technique is applied to simple gamma-ray tomography algorithms. Furthermore, the impact of these regularization techniques on the solutions of linear systems of equations is significant. An ART (Algebraic Reconstruction Technique)-type algorithm was used for the reconstruction of simulated data utilizing built-in Matlab functions. These were compared with data obtained through a regularization implemented with TSVD (Truncated Singular Value Decomposition), as well as data obtained through hybrid algorithms such as TSVD plus Toepelitz, tridiagonal and identity operators. The quality of the resulting reconstruction is evaluated through RMSE (Root Mean Square Error). Direct comparisons suggest that for a high noise level and high delta parameter the TSVD plus tridiagonal operator is the best choice.

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1. Introduction

Tomography has since the 1930s been used for nondestructive evaluation of materials in different areas, including medical and industrial applications. This evaluation is performed by inspecting a representation of the object's density distribution function in the interior of its vessel's section, usually an image representation, which takes advantage of the relation that exists between the density distribution and the attenuation or perturbation suffered by any penetrating wave, such as magnetic resonance, ultrasound, electrical capacitance, radioactive emission, seismic, etc. This process of image reconstruction is modeled as an inversion of the Radon Transform, which basically represents the physical experiment of producing attenuation values out of beams crossing the vessel plus object. There are a number of reconstruction methods divided in different types, such as analytical, algebraic, iterative and statistical (see Gordon et al., 1970; Kak and Slaney, 2001–2007; Maad et al., 2008; Maad, 2009; Melo et al., 2007). It is well known that this type of inverse problem is ill-posed in the Hadamard's sense (see Hadamard, 1902), for which the solution for a well posed problem should exist, be unique and continuously dependent on the initial data (stability). Inverting Radon Transform becomes even more ill-posed as the number of views and beams per view decreases. In addition, there is a certain amount of error real experiments which are prone to. The reconstruction results in this scenario, which is even harsher in the industrial applications, end up presenting an error level which is unbounded even when the input error is bounded. The conventional solution to this has been an image pre- or post-processing, but superior results have been achieved through regularization, that consists in finding an approximation to the density distribution, which is the solution to a similar problem where the output error is continuously dependent

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on the input error. There are several types of regularization, which have been applied to different types of tomography modalities: EIT (Electrical Impedance Tomography): Hua et al. (1991), Binley et al. (1995), Adler and Guardo (1996), Vauhkonen et al. (1998), Stephenson et al. (2005), Karsten and Helmut (2005), Borsic et al. (2010); ECT (Emission Computed Tomography): Lee (2003), Burger et al. (2014); X-ray Tomography: Titarenko et al. (2010); and Gamma-ray Tomography: Azzi et al. (1991), Dantas et al. (2008), among others.

In a previous work (Santos et al., 2011), relevant gammametric information from a gas–solid flow in a riser was extracted out of raw noisy data through a PCA (Principal Component Analysis) method. In this work we investigate the impact of a Tikhonov regularization on the solution of a gamma-ray tomography reconstruction by means of a least squares numerical method. An image reconstruction based on algebraic computational algorithm (ART – Gordon et al., 1970) involving sparse under- and over- determined linear system of equations is analyzed. Built-in functions of Matlab software were applied and optimal solutions were investigated. Experimentally a section of the tube is scanned from various positions and at different angles. The solution, to find the coefficients vector μ, from the vector of measured p values through the W matrix inversion, constitutes an inverse problem. Such a solution is often required in industrial process tomography in presence of experimental noise and aiming at a shorter time resolution. The definition of Hadmard's inverse problem is considered, as well as the requirement of a well posed problem to find stable solutions. The formulation of the basis function and the algorithm used to structure the weight matrix are discussed. For a full rank weight matrix, the obtained solution is unique, as expected. In a previous work by Araújo et al. (2009), the stability of the solution was investigated by means of a Tikhonov regularization technique and the results obtained for the inverse problem solution were quite promising. The results showing a significant improvement on image reconstruction were obtained with an ART algorithm in a single beam tomography. Nevertheless, a better understanding of the noise level effect requires a regularization technique associated to Morozov's principle. In this work, a further theoretical formulation of the regularization is given and the solutions for the inverse problem, using computational algorithms, are discussed.

2. Methodology

To reconstruct an image and produce a graphical representation of the distribution of the process parameters, it is necessary to limit the spatial resolution and define an attenuation function μ. This is an array of pixels, for which each linear attenuation coefficient is assumed to be constant. The ray sum for each ray j is then expressed by the following summation:

$$p_j = \sum_{i=1}^{N} w_{ji}\mu_i \tag{1}$$

where w_{ji} is the contribution to ray sum j from pixel i, μ_i is the attenuation coefficient of the pixel i and N is the total amount of pixels. The terms w_{ji} constitute the entries of the weight matrix W. Assuming that we have m measurements p_j ($j = 1, \ldots, m$), where m is usually bigger than the number of pixels in μ, we obtain the following system of linear equations:

$$W\mu = p \tag{2}$$

The solution of the matrix Equation (2) requires the inversion of the weight matrix W to find μ according to a given density vector p; it constitutes the inverse problem in tomography. Since W usually is not a square matrix, the solution of the least squares problem to find the μ vector will minimize the norm of the residuals via normal equations. An ART-type algorithm (Algebraic Reconstruction Technique) was developed in the Matlab environment and the solution of the linear system of equations as it is given in Equation (2) was investigated. The Matlab built-in functions include several known matrix factorization algorithms. Its linear solvers can produce an $m \times n$ W matrix with either $m < n$, $m = n$ or $m > n$, according to the available tomographic arrangement. For underdetermined systems ($m < n$), nearly singular matrix and rank deficient matrix can bring perturbations to the least squares and even prevent it from reaching a solution. The investigation of such a solution deals with the very nature of an inverse problem. If the coefficient matrix of the linear system in Equation (2) is invertible and well conditioned, then the vector of coefficients μ can be obtained from the vector of measured p values as:

$$\mu = W^{-1}p \tag{3}$$

where W is a given mapping (operator) from a space U into a space P; $\mu \in U$ is to be given from a $p \in P$. Hadamard's well-posedness conditions, Hadamard (1902) are as follows: a solution should exist, should be unique and be continuously dependent on the initial data. Applied to equation (3) Hadamard's conditions take the form:

i) for any $p \in P$, there exists $\mu \in U$ such that (1) holds, i. e., $R(W) = P$ where $R(W)$ is the range of values of W (existence).
ii) The data p determines the solution μ uniquely, i.e., there exists an inverse operator W^{-1} (uniqueness).
iii) The solution μ depends continuously on the data p, i.e., the inverse operator W^{-1} is continuous (stability).

If Equation (3) satisfies these requirements, the problem is said to be well-posed on the pair of topological spaces U, P. Otherwise, the problem is *ill-posed*. In industrial gamma-ray tomography there are serious restrictions on the amount of transmitting beams, due to layout, security and equipment cost limits, causing the acquired data to be much noisier than in the medical counterpart, which calls for the linear system inversion to be done with a regularization, as proposed in a well known paper by Azzi et al. (1991).

The problem of regularization, in general, is equivalent to the Lagrange multiplier problem of determining $\lambda > 0$ such that

$$\left(W^{\mathrm{T}}W + \lambda I\right)x = W^{\mathrm{T}}b \tag{4}$$

and $||x|| = \alpha$. This equation is precisely the normal equation formulation for the ridge regression problem

$$\min_x \left\| \begin{bmatrix} W \\ \sqrt{\lambda} I \end{bmatrix} x - \begin{bmatrix} b \\ 0 \end{bmatrix} \right\|_2^2 = \min_x \|Wx - b\|_2^2 + \lambda\|x\|_2^2 \tag{5}$$

which was proposed first by Tikhonov (1963). The left side is basically the stacking of the two terms on the right, while Equation (4) can be seen as the evaluation of the Euclidean norm in Equation (5) and the determination of the minimum through differentiation. The idea behind this equation is the attempt to attack the numerical instability (item *iii* earlier) by minimizing not only the residuals but also the norm of the solution. In the general ridge regression problem one has some criteria for selecting the ridge parameter λ and I matrix of additional information, e.g., $\|x(\lambda)\|_2 = \alpha$, for some given α.