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An analytical solution of the point kinetics equations with time-variable reactivity by the decomposition method

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ABSTRACT

In this work, we report an analytical solution for the point kinetics equations by the decomposition method, assuming that the reactivity is an arbitrary function of time. The main idea initially consists in the determination of the point kinetics equations solution with constant reactivity by just using the well-known solution results of the first-order system of linear differential equations in matrix form with constant matrix entries. Applying the decomposition method, we are able to transform the point kinetics equations with time-variable reactivity into a set of recursive problems similar to the point kinetics equations with constant reactivity, which can be straightly solved by the mentioned technique. For illustration, we also report simulations for constant, linear and sinusoidal reactivity time functions as well comparisons with results in literature.

1. Introduction

Lately, an increasing interest in the task of searching for analytical solutions of linear and nonlinear problems by the scientific community has been noticed. Besides the mathematical elegance, the analytical solution possesses a relevant aptness to generate benchmark solutions to validate computational code results (Ganapol, 2008). Further, the analytical solution in some sense, eliminates or at least mitigates the difficulty of the mathematical task of the error evaluation required by numerical methods, except for the round-off error. We must mention that by analytical we mean that no approximation is done along the solution derivation. For illustration of the literature about analytical solution, we cite the works of Vilhena et al. (2008a,b), Tirabassi et al. (2008), Goulart et al. (2008) and Carvalho et al. (2007).

In this work, keeping in the track of searching for analytical solutions, we apply the decomposition method (Adomian, 1994) to solve the point kinetics equations for a time-variable reactivity. The main idea comprehends the following steps: a) expansion of the neutron density and delay neutron concentrations in a truncated series, b) replacement of these *ansatz* in the point kinetics equations and construction of a set of recursive systems of first-order differential equations similar to the point kinetics equations with constant reactivity. The recursive solution of the system is promptly obtained recasting these equations into matrix form and using the well-known solutions for a system of first-order linear differential equations with constant matrix entries. We are aware of the works in literature concerning the solution of the point kinetics equations with time-variable reactivity using numerical approaches, among them we mention the works of Hansen et al. (1965), Nobrega (1971), Kueng (1977), Chao and Attard (1985), Sanchez (1989), Aboanber and Nahla (2002) and Kinard and Allen (2004), however it is still worth trying to obtain a general analytical approach of any type of time-varying reactivity. To enlighten the feasibility of the proposed methodology to solve this sort of problem, we report numerical simulations and convergence analysis as well comparisons with available results in literature for constant, linear and sinusoidal time-variable reactivity.

2. Analytical solution of the point kinetics equations

In order to construct the solution of the point kinetics equations for variable reactivity, let us consider the model assuming six delayed neutron families:

$$\begin{cases} \dfrac{d}{dt}n(t) = \dfrac{\rho(t)-\beta}{\Lambda}n(t) + \displaystyle\sum_{i=1}^{6}\lambda_i C_i(t), \\ \dfrac{d}{dt}C_i(t) = \dfrac{\beta_i}{\Lambda}n(t) - \lambda_i C_i(t), \quad i = 1,\dots,6, \end{cases} \tag{1}$$

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with the following initial condition:

$$\begin{cases} n(0) = n_0, \\ C_i(0) = \frac{\beta_i}{\lambda_i \Lambda}, \end{cases} \tag{2}$$

for $i = 1,\ldots,6$. Here $n(t)$ denotes the neutron density, $\rho(t)$ is the time-variable reactivity, β_i is the fraction of fission neutrons emitted by the decay of the ith precursor, characterized by effective concentration $C_i(t)$, β is the total delay fraction, Λ is the effective mean prompt generation time, λ_i is the decay constant. In order to apply the decomposition method, we expand the neutron and delayed neutron concentration in a truncated series, as:

$$\begin{aligned} n(t) &= \sum_{j=1}^{N} n_j(t), \\ C_i(t) &= \sum_{j=1}^{N} C_{ij}(t), \end{aligned} \tag{3}$$

for $i = 1,\ldots,6$. The replacement of the expansions (3) in the point kinetics equations (1) leads to:

$$\begin{cases} \frac{d}{dt}(n_0 + n_1 + \cdots n_N) = \left(\frac{\rho_0 + \rho_1(t) - \beta}{\Lambda}\right)(n_0 + n_1 + \cdots n_N) + \sum_{i=1}^{6} \lambda_i (C_{i0} + C_{i1} + \cdots C_{iN}), \\ \frac{d}{dt}(C_{i0} + C_{i1} + \cdots C_{iN}) = \frac{\beta_i}{\Lambda}(n_0 + n_1 + \cdots n_N) - \lambda_i (C_{i0} + C_{i1} + \cdots C_{iN}), \quad i = 1, \ldots, 6, \end{cases} \tag{4}$$

where the explicit dependence on time has been omitted for simplicity. The reactivity ρ is written as $\rho = \rho_0 + \rho_1(t)$, where ρ_0 is a constant value. We must notice that this system is undetermined because $7 \times N$ unknowns exist for only seven equations. We overcome this difficulty by constructing the following recursive system:

$$\begin{cases} \frac{d}{dt} n_0(t) = \frac{\rho_0 - \beta}{\Lambda} n_0(t) + \sum_{i=1}^{6} \lambda_i C_{i0}(t), \\ \frac{d}{dt} C_{i0}(t) = \frac{\beta_i}{\Lambda} n_0(t) - \lambda_i C_{i0}(t), \quad i = 1, \ldots, 6, \end{cases} \tag{5}$$

for the first term of the solution expansion and

$$\begin{cases} \frac{d}{dt} n_j(t) = \frac{\rho_0 - \beta}{\Lambda} n_j(t) + \sum_{i=1}^{6} \lambda_i C_{ij}(t) + \frac{\rho_1(t)}{\Lambda} n_{j-1}(t), \\ \frac{d}{dt} C_{ij}(t) = \frac{\beta_i}{\Lambda} n_j(t) - \lambda_i C_{ij}(t), \quad i = 1, \ldots, 6, \end{cases} \tag{6}$$

for the generic term of the solution expansion, with $j = 1,\ldots,N$. Here we need to remark that although not unique this decomposition is done in such a manner that the fulfillment of equation (4) is guaranteed. Recasting this recursive system into matrix form, we have:

$$\begin{pmatrix} \frac{d}{dt} n_0 \\ \frac{d}{dt} C_{10} \\ \vdots \\ \frac{d}{dt} C_{60} \end{pmatrix} = \begin{pmatrix} \frac{\rho_0 - \beta}{\Lambda} & \lambda_1 & \cdots & \lambda_6 \\ \frac{\beta_1}{\Lambda} & -\lambda_1 & 0 & 0 \\ \vdots & 0 & \ddots & \vdots \\ \frac{\beta_6}{\Lambda} & 0 & \cdots & -\lambda_6 \end{pmatrix} \begin{pmatrix} n_0 \\ C_{10} \\ \vdots \\ C_{60} \end{pmatrix} \tag{7}$$

for the homogeneous equation and

$$\begin{pmatrix} \frac{d}{dt} n_j \\ \frac{d}{dt} C_{1j} \\ \vdots \\ \frac{d}{dt} C_{6j} \end{pmatrix} = \begin{pmatrix} \frac{\rho_0 - \beta}{\Lambda} & \lambda_1 & \cdots & \lambda_6 \\ \frac{\beta_1}{\Lambda} & -\lambda_1 & 0 & 0 \\ \vdots & 0 & \ddots & \vdots \\ \frac{\beta_6}{\Lambda} & 0 & \cdots & -\lambda_6 \end{pmatrix} \begin{pmatrix} n_j \\ C_{1j} \\ \vdots \\ C_{6j} \end{pmatrix} + \begin{pmatrix} \frac{\rho_1(t)}{\Lambda} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} n_{j-1} \\ C_{1j-1} \\ \vdots \\ C_{6j-1} \end{pmatrix} \tag{8}$$

for the non-homogeneous equations, with $j = 1,\ldots,N$. To this point, we must underline that the solution of equations (7) and (8) is straightly determined by the well-known results of first-order linear differential matrix equations. In fact, let us consider the system of first-order linear differential equations in matrix form:

$$\frac{dY}{dt} - AY = H(t), \tag{9}$$

where Y is the unknown column vector, H is the source vector and A is a matrix with constant entries. The well-known solution for matrix equation (9) reads as:

$$Y(t) = \exp(At)Y(0) + \int_0^t \exp(A\tau)H(t-\tau)d\tau. \tag{10}$$

On the other hand, for the special problems in which the eigenvalues of the matrix A are distinct, the exponential of matrix A is expressed by:

$$\exp(At) = X \exp(Dt) X^{-1}, \tag{11}$$

where X is the matrix of the eigenvectors of A and X^{-1} is its inverse. The matrix D is the diagonal matrix whose elements are the eigenvalues of the matrix A. Bearing in mind that the eigenvalues of the matrix appearing in equations (7) and (8) are distinct, the solution of problem (1) is well determined by equation (3), recalling

Table 1A
Comparisons of solution for the power with different values of Λ and for a constant reactivity $\rho(t) = \rho_0$. Results obtained by DM. Results above 1×10^{10} are not shown.

Λ [s]	Time [s]	$\rho_0 = 0.003$	$\rho_0 = 0.007$	$\rho_0 = 0.008$
1×10^{-3}	1×10^{-5}	1.00003	1.00007	1.00008
	1×10^{-3}	1.00299	1.00700	1.00800
	1×10^{-1}	1.24847	1.70343	1.84549
	1	1.96728	11.0252	20.7376
	10	6.65661	1.41405×10^6	1.15587×10^9
1×10^{-4}	1×10^{-5}	1.0003	1.00070	1.00080
	1×10^{-3}	1.02941	1.07000	1.08040
	1×10^{-1}	1.76528	8.34743	15.4198
	1	2.18765	1055.51	1.14917×10^6
	10	7.88318	–	–
1×10^{-5}	1×10^{-5}	1.00299	1.00700	1.00800
	1×10^{-3}	1.24727	1.70004	1.84141
	1×10^{-1}	1.80228	110.757	2,229,680
	1	2.21266	4.0266×10^8	–
	10	8.03659	–	–
1×10^{-6}	1×10^{-5}	1.02941	1.07000	1.080400
	1×10^{-3}	1.73656	8.00355	14.753100
	1×10^{-1}	1.80477	15,068.9	–
	1	2.21520	–	–
	10	8.05232	–	–