Journal of Petroleum Science and Engineering 133 (2015) 740–743

Contents lists available at ScienceDirect

Journal of Petroleum Science and Engineering

journal homepage: www.elsevier.com/locate/petrol

On the computation of an integral of the modified Bessel function, $\int_x^\infty d\lambda\ \lambda^\mu K_\nu(\lambda)$

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ARTICLE INFO

Article history:
Received 22 August 2013
Received in revised form
12 July 2015
Accepted 14 July 2015
Available online 17 July 2015

Keywords:
Modified K Bessel functions
Integrals of Bessel functions
Bickley functions
Anomalous diffusion
Fractured rocks
Fractured wells

ABSTRACT

The work introduces a simple numerical scheme to evaluate the integral $\int_x^\infty d\lambda\ \lambda^\mu K_\nu(\lambda)$ where $K_\nu(\lambda)$ is the modified Bessel function of fractional order. We use this scheme to compute the response at a fractured well that describes transient diffusion in porous media considered to be fractal structures.

1. Introduction

The long-standing interest in computations of well responses of fractured wells involves the evaluation of the integral $\int_x^\infty d\lambda\ K_0(\lambda)$, where $K_0(x)$ is the modified Bessel function of zero order as noted in Ozkan and Raghavan (1991a,b). Their method, particularly useful for addressing both infinite- and finite-conductivity fractures for classical diffusion, supplanted time-domain methods such as those proposed in Gringarten et al. (1974) primarily because it may be readily adapted to address variable-rate problems and to incorporate naturally fractured reservoir models among other advantages. For anomalous diffusion, that is, for diffusion in rocks that may be considered to be fractal-like structures we have relied on the model of Beier (1994) which is a time-domain method. The solution to the Beier formulation that is an analog of the Ozkan and Raghavan (1991a,b) development involves the evaluation of $\int_x^\infty d\lambda\ \lambda^\mu K_\nu(\lambda)$ where $K_\nu(x)$ is the modified Bessel function of fractional order, ν, as noted in Raghavan and Chen (2013). The purpose of this note is to provide a commentary on the computation of this integral.

Although there appears to be a long-standing interest in the computation of the integral under consideration as noted in Luke (1962), we found his suggestions to be of benefit primarily for computing early-time responses particularly when used in conjunction with the numerical computation of the Laplace transformation by methods such as Stehfest (1970a,b). To our knowledge, the work of others concerns the computation of this integral only for $\mu = 0$ such as Kostroun (1980) whose interest in such an integral was in the context of synchrotron radiation and indirectly by Rothman (1954) because of his interest in evaluating the integral of the Airy function, $Ai(x)$, that is related to $K_{1/3}(x)$; see Abramowitz and Stegun (1972). Closed form expressions for $\int_0^\infty d\lambda\ \lambda^\mu K_\nu(\lambda)$ may be found in Luke (1962) and in Spanier and Oldham (1987). As of yet, a robust computational scheme to compute $\int_x^\infty d\lambda\ \lambda^\mu K_\nu(\lambda)$ has been unavailable.

2. A series expression for $\int_x^\infty d\lambda\ \lambda^\mu K_\nu(\lambda)$

Starting with the integral representation of Schläfli given in Watson (1922) for $K_\nu(z)$ given by

$$K_\nu(z) = \int_0^\infty d\chi\ \exp(-z\ \cosh\ \chi)\cosh\ \nu\chi; \quad |\arg z| < \frac{\pi}{2}, \tag{1}$$

and using

$$\int_x^\infty d\lambda\ \lambda^\mu\ \exp(-\lambda\ \cosh\ \chi) = \frac{1}{(\cosh\ \chi)^{\mu+1}}\Gamma(\mu+1, x\ \cosh\ \chi), \tag{2}$$

where $\Gamma(a; x)$ is the *complementary incomplete Gamma function*, we may write

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http://dx.doi.org/10.1016/j.petrol.2015.07.017
0920-4105/

Nomenclature

a	site density parameter of fractal structure, L^{-d_f}
c	compressibility L T^2
d	dimension, 1, 2 or 3
d_f	fractal (Hausdorff) dimension
d_w	anomalous diffusion coefficient (random walk dimension)
$K_\nu(z)$	modified Bessel function of order ν
L_f	fracture half-length [L]
ℓ	reference length [L]
m	fracture network parameter in a fractal system [$L^{\theta+2}$]
p	pressure [M/L/T^2]
p'	logarithmic derivative [M/L/T^2]
q	rate [L^3/T]
$\mathbb{R}$	Real part of a number
r	distance [L]
t	time [T]
V_s	volume per site of fractal structure [L^3]
$\Gamma(x)$	gamma function
$\Gamma(a;x)$	complementary incomplete gamma function
θ	scaling variable
μ	exponent
ν	order of modified Bessel function

Subscripts

D	dimensionless
f	fractal dimension

Superscript

–	Laplace transform

$$\int_x^\infty d\lambda\ \lambda^\mu K_\nu(\lambda) = \int_0^\infty d\chi \Gamma(\mu+1; x\ \cosh\ \chi)\frac{\cosh\ \nu\chi}{(\cosh\ \chi)^{\mu+1}}. \tag{3}$$

This expression is similar to the Bickley (1935) function, $Ki_1(x)$, used by Ozkan and Raghavan (1991), namely,

$$Ki_\alpha(x) = \frac{1}{\Gamma(\alpha)}\int_x^\infty d\lambda(\lambda-x)^{\alpha-1}K_0(\lambda)$$

$$= \int_0^\infty d\chi\frac{e^{-x\ \cosh\ \chi}}{(\cosh\ \chi)^{\alpha+1}};\quad x>0. \tag{4}$$

Further, we arrive at the expression of Kostroun (1980) from (3):

$$\int_x^\infty d\lambda\ K_\nu(\lambda) = \int_0^\infty d\chi\ \exp(-x\ \cosh\ \chi)\frac{\cosh\ \nu\chi}{\cosh\ \chi}, \tag{5}$$

if we use

$$\Gamma(\nu+1;x) = \nu\Gamma(\nu;x) + x^\nu\ \exp(-x). \tag{6}$$

Any scheme for quadrature may be used to evaluate the integral in (3). For our purposes, like Kostroun (1980), we have found the trapezoidal rule to be adequate, namely,

$$\int_z^\infty d\lambda\ \lambda^\mu K_\nu(\lambda) = h\left\{\frac{\Gamma(\mu+1;z)}{2} + \sum_{r=1}^\infty \Gamma[\mu+1;z\ \cosh(rh)]\frac{\cosh(\nu rh)}{(\cosh\ rh)^{\mu+1}}\right\}, \tag{7}$$

where $h/2\pi < 1$.

Sample results to illustrate the scheme in (3) for a few values of x, ν and μ are given in Tables 1 and 2, for a tolerance, ϵ, of 10^{-5}. Many of the results in Table 1 are for a specific relationship between μ and ν because analytical expressions that are given below are available for comparison. Results in Table 2 are for $\mu=0$ and $\nu = 1/3$ (the integral of the Airy function) while the results of Table 3 are for $\mu=-1$.

Column 5 of Table 1 notes values obtained through the trapezoidal rule; the number of terms used, r, is also noted. Results in Column 6 are obtained from the expressions given in (8) or (9) taken from Luke (1962, p. 126), namely,

$$\int_0^x d\lambda\ \lambda^\nu K_{\nu-1}(\lambda) = 2^{\nu-1}\Gamma(\nu) - x^\nu K_\nu(x);\quad \mathbb{R}(\nu)>0, \tag{8}$$

and

Table 1
Sample results, series expansion, (7), $\mu \neq 0$; $h = 0.5$.

x	ν	μ	r	(7)	(8) or (9)	(10)
0.01	−0.99	0.01	18	45.551335	45.551335	
1			9	49.641057	49.641057	49.532872
7			5	50.061664	50.061664	50.061688
0.01	−0.01	0.99	17	0.000275	0.000275	
10			5	0.998742	0.998742	0.998744
15			4	0.998922	0.998922	0.998922
0.01	0	1	17	0.000261	0.000261	
1			9	0.398093	0.398093	
10			5	0.999813	0.999814	0.999844
0.01	1	2	18	4.96×10^{-5}	4.88×10^{-5}	
1			9	0.375161	0.375161	
10			5	1.997848	1.997850	1.997979
0.01	−0.99	0.1	19	5.471834		
5			12	9.229874		9.230215
0.01	0.01	0.1	35	0.032316		
1			17	1.053667		1.020091
5			12	1.395738		1.395774

Table 2
Integral of the Airy function.

x	r	**A & S**, Table 10.12	(7)
0.1	13	0.0342101	0.0342101
1.0	6	0.2363173	0.2363173
5.0	2	0.3332876	0.3332882
7.5	2	0.3333333	0.3333326

Table 3
The function $x\exp(x)\int_x^\infty d\lambda\ K_0(\lambda)/\lambda$.

x	r	**A & S**, Table 11.2	(7)
0.1	13	0.368126	0.368126
1.0	9	0.566811	0.566811
5.0	8	0.534635	0.534635

$$\int_x^\infty d\lambda\ \lambda^{-\nu}K_{\nu+1}(\lambda) = x^{-\nu}K_\nu(x). \tag{9}$$

The last column in Table 1 is a tabulation of results through the asymptotic expansion of the integral for large z given in Luke