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Trapped modes in an infinite or semi-infinite tube with a local enlargement

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ABSTRACT

Trapped modes in a hard cylindrical tube with a local axisymmetric enlargement or bulge and filled with a uniform acoustic medium is studied. The governing Helmholtz equation in the cylindrical coordinate system is employed to deal with this problem through the domain decomposition method and matching technique. The trapped modes and the corresponding frequencies less than the threshold frequency or cut-off frequency are derived. It is found that in addition to the fundamental mode, the second- and higher-order trapped modes exist and depend on the geometry parameters of the local bulge. The effects of the bulge radius and width on the frequencies are discussed. The local bulge leads to a decrease of the frequencies and the corresponding vibration mode is localized near the bulge. A multimodal analysis is made and frequency band gap of generalized trapped modes is also studied. A frequency band gap depends on the radius of a bulge and is independent of its width. The obtained results can be extended to analyze bound states in quantum wires.

1. Introduction

Trapped modes refer to localized time-harmonic oscillations of finite energy within a medium which is unbounded in at least one direction. Trapped modes are observed to exist in many fields due to various causes. For example, for a two-dimensional acoustical waveguide, Callan et al. [1] studied trapped modes for an infinitely long strip containing a circle. Acoustic resonances and trapped modes in an infinitely long cylindrical tube with a sound-hard sphere were further formulated [2,3]. In addition, Martin [4,5] considered acoustic waves in a rigid axisymmetric tube with a variable cross-section. The essence of such problems is to solve eigenstates of the Helmholtz equation subject to appropriate boundary conditions. For this kind of problems, there are a wide application.

In quantum waveguides, trapped modes are called bound states. The Helmholtz equation is equivalent to the time-independent Schrodinger equation. Using finite differences in a truncated finite domain, Schult et al. [6] first studied the bound states in an unbound system of crossed wires and calculated the energy and wavefunction for an electron at the intersection of two symmetric narrow regions in a two-dimensional plane. Later, Amore et al. [7] extended the above approach to treat asymmetric cross, T- and L-shaped configurations, and found that except for the even-even case, there are other odd–odd and even–odd bound states of an electron localized at the intersection of two wires under some geometry constraints. Delitsyn et al. [8] further utilized the variational approach to cope with trapped modes in finite quantum waveguides. Londergan and Murdock [9] compared several methods to solve the trapped mode problems or quantum bound states for two-dimensional pipes. Andrews and Savage [10] applied the conformal mapping to analyze bound states for an infinite strip with a smooth bulge. Ordonez et al. [11] used the integral equation method to investigate bound states for two open quantum dots connected by a wire in the two-dimensional plane. Avishai et al. [12] studied the existence of bound states of two-dimensional Helmholtz equations with Dirichlet boundary conditions in open domains. Furthermore, based on the approach of the wave equation and appropriate boundary conditions, Helie [13] formulated a model of the linear acoustic propagation in axisymmetric waveguides. A full multimodal analysis of a two-dimensional open rectangular-shaped groove waveguide has been made for both TE and TM modes using a Galerkin approximation [14]. For two parallel waveguides of different widths in a two-dimensional plane, bound states can occur in coupled quantum wires through a finite length window [15]. The analysis is further extended to three-dimensional case: two concentric circular cylindrical waveguides coupled by a finite length gap along the axis of the inner cylinder [16]. Later, for a slowly varying cross section,

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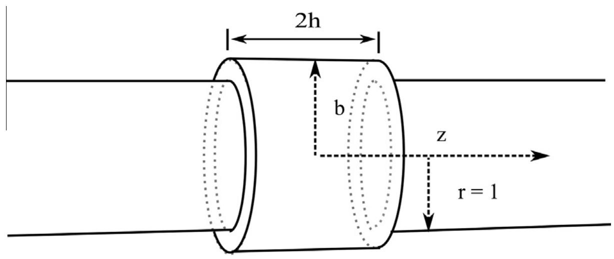

Fig. 1. An infinite tube with a local axisymmetric enlargement.

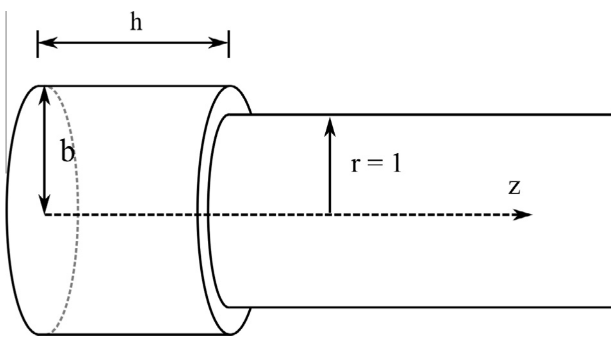

Fig. 2. A semi-infinite tube with an end axisymmetric enlargement.

Gaulter and Biggs [17] investigated acoustic trapped modes through asymptotic approximation method.

Recently, Wang [18] presented the domain decomposition method and matching technique to solve trapped modes for a membrane strip with a local enlargement in the two-dimensional plane. Furthermore, the method is extended to study vibration of a membrane strip with a segment of higher density and trapped modes were also found due to inhomogeneity [19].

This paper is focused on waveguide of an infinite or semi-infinite tube. It is found that trapped modes due to the presence of a local bulge exist near the bulge. The frequencies of trapped modes are numerically determined by the domain decomposition and matching technique and they are less than the threshold or cut-off frequency of a uniform tube without any bulge. In particular, in addition to the fundamental mode, the second- or higher-order trapped modes exist.

2. Infinite tube with a local enlargement

Consider the waveguide in an infinite hard cylindrical tube of radius R, filled with a linear uniform acoustic medium, in which a local enlargement of radius $R^*(R^* > R)$ with finite width $2H$ appears in the tube, as shown in Fig. 1. A cylindrical polar coordinate system is chosen such that the z-axis is orientated in the centroid line or longitudinal axis of the tube. When acoustic waves travelling in the tube, only axisymmetric excitations are considered so that the whole problem is axisymmetric and velocity potential ϕ considered here is independent of an angular variation and, under the assumption of irrotational motion with small amplitude, satisfies the following governing equation

$$\frac{\partial^2\phi}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial\phi}{\partial\rho}+\frac{\partial^2\phi}{\partial Z^2}=\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}, \tag{1}$$

where c is the speed of sound. When denoting the velocity potential as $\phi = \Phi(\rho, Z)e^{i\omega t}$, ω being angular frequency, the above governing equation is reduced to the Helmholtz equation

$$\frac{\partial^2\Phi}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial\Phi}{\partial\rho}+\frac{\partial^2\Phi}{\partial Z^2}+\frac{\omega^2}{c^2}\Phi=0. \tag{2}$$

As we know, acoustic waves propagating along an infinitely long tube must have a minimum frequency as the fundamental frequency and those with frequency in excess of the fundamental frequency can travel. However, if a tube has a local enlargement, the fundamental frequency is reduced, as will be seen, and local vibration of trapped modes occurs near the local enlargement. In order to make it easier to analyze local vibration near the enlargement zone, let us introduce dimensionless variables as follows

$$w=\frac{\Phi}{R},\quad r=\frac{\rho}{R},\quad z=\frac{Z}{R},\quad k=\frac{\omega R}{c}. \tag{3}$$

Thus Eq. (2) is rewritten as a normalized form

$$\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{\partial^2 w}{\partial z^2}+k^2 w=0. \tag{4}$$

For an infinitely long tube, the one-dimensional solution to Eq. (4) is easily found by neglecting z, and it is $w = J_0(kr)$ with the requirement of a finite value at $r = 0$, where $J_0(*)$ is the zeroth order Bessel function of the first kind. Furthermore, due to the hard tube wall, we have vanishing Dirichlet boundary condition at $r = 1$, the fundamental frequency parameter k is obtained to be $\lambda_1 = 2.4048$ as the lowest positive value such that $J_0(k) = 0$ and the corresponding eigenfunction is $J_0(\lambda_1 r)$. In other words, velocity potential in the cylindrical waveguide has a threshold value or cut-off frequency $\lambda_1 = 2.4048$. Similarly, if removing λ_1 and other natural frequencies such as $\lambda_j (j = 2, 3, \ldots,)$ as threshold value of cut-off frequency, these modes mean generalized trapped modes. In fact, for the latter, the frequencies of propagating modes may also lie in the region.

Our task is to seek trapped modes, i.e. localized time-harmonic oscillations of finite energy occur near the local enlargement of the tube. Since our attention is focused on the local vibration near the local enlargement, we denote the tube domain without bulge as $|z| \geqslant h$, $r \leq 1$ (Region I), and the bulge segment as $|z| \leqslant h, r \leqslant b$ (Region II), where $2h$ and $2b$ designate the corresponding dimensionless width and diameter of the bulge segment, respectively,

$$h=\frac{H}{R},\quad b=\frac{R^*}{R}. \tag{5}$$

Table 1
Convergence rate of k_1 for an infinite tube with a local axisymmetric bulge with $h = 1$.

N	2	6	10	20	40	60
$b = 2$	1.7653	1.7734	1.7751	1.7762	1.7767	1.7767
$b = 10$	1.5522	1.5548	1.5572	1.5584	1.5591	1.5591

Table 2
Frequency parameter k_j of trapped modes for an infinite tube with a local axisymmetric bulge with $h = 1$.

b	1	1.5	2	4	8	10	20	40
k_1	2.4048	1.9743	1.7767	1.5953	1.5616	1.5591	1.5570	1.5569
k_2	–	–	–	1.9926	1.6721	1.6326	1.5829	1.5732
k_3	–	–	–	–	1.8592	1.7564	1.6143	1.5805