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Analyzing modal behavior of guided waves using high order eigenvalue derivatives

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ABSTRACT

This paper presents a mode-tracing approach for elastic guided waves based on analytically computed derivatives and includes a study of interesting phenomena in the dispersion curve representation. Numerical simulation is done by means of the Scaled Boundary Finite Element Method (SBFEM). Two approaches are used to identify the characteristics of the resulting wave modes: Taylor approximation and Padé approximation. Higher order differentials of the underlying eigenvalue problem are the basis for these approaches. Remarkable phenomena in potentially critical frequency regions are identified and the tracing approach is adapted to these regions. Additionally, a stabilization of the solution process is suggested.

1. Introduction

Simulations and modeling in material science and research have significantly increased in value over the past decades. As new methods arise and higher precisions in measurements are possible, the demand for simulations in several fields has grown steadily. Although computational capabilities have developed to impressive dimensions, the demand for computational power of classical simulations has exceeded these capabilities. Therefore, new problem specific simulation tools are optimized for many fields and problems. A limitation or purpose-built way of usage is usually the consequence of this optimization. A common approach is to optimize the solution process by fixing a parameter, meaning that the system is solved efficiently, but only for one given value of this parameter. Tracing techniques attempt to transform these fixed parameter approaches into continuous solution sets without losing computational advantages.

The propagation of elastic guided waves is a field with increasing need for simulations. The significance of this field is shown by its range of applications such as Non-destructive testing [1–3], structural health monitoring [4] and material characterization [5]. Thus, an increasing number of experiments and publications on guided waves demand simulations with high complexity. The representation of the time harmonic wave propagation of guided waves with dispersion curves is crucial for the high complexity of the problem at hand, i.e. the frequency-dependent phase and group velocities of guided waves in given structures and the corresponding mode shapes. The computation of these dispersion curves works with high frequencies and steadily increasing dimensions of geometries, which easily explains the high computational complexity. Additionally many applications solve inverse problems and therefore have to perform these complex simulations repeatedly.

Simulation with sufficient accuracy is, therefore, a challenging task. In principle, established Finite Element Methods (FEM) are able to model and simulate wave propagation and its phenomena. The FEM is a powerful tool which works effectively for a number of purposes, however, it is not designed for this specific problem. Small wavelengths in large geometries require a highly extensive number of degrees of freedom and therefore increase computational costs significantly. Even more so, the high frequencies ask for small time steps in transient simulations. For the case of repeated simulations, especially in the solution process of inverse problems, the computational costs are often not acceptable.

Several approaches have been suggested in order to increase efficiency while preserving accuracy. Especially the case of waveguides with long homogeneous sections is of high interest as it appears regularly in many applications. Using this rather simple property, computational time can be decreased significantly without losing accuracy.

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Analytical approaches are well known and highly efficient for certain structures. Solution procedures to compute dispersion relations and mode shapes have been developed for infinite plates and infinite cylinders [6–9]. These procedures allow for better analysis of signals or extraction of model reflection coefficients from standard Finite Element analysis. This extraction allows for efficient simulation of interactions with cracks [12] or notches [10,11,13], thereby rendering it a powerful tool for said specific geometries.

Semi-analytical methods allow for the inclusion of more general geometries or inhomogeneous materials in infinite waveguides. In essence, these methods describe extended sections of the model analytically and the remains numerically. An early approach of Kausel [14,15] was one of the first formulations by using linear interpolation in the thickness direction of plates, known as the Thin Layer Method (TLM). The resulting quadratic eigenvalue problem computes wave numbers in plates or soil layers and efficiently simulates wave propagation in soil [16,17], anisotropic plates [18] and even piezo composite layers [19]. Furthermore, this method allows for analytical computation of stiffness elements [20]. Broader numerical insights can be found in [21].

For the specific case of ultrasonic guided waves in solids, the concept of discretizing the cross-section and applying an analytical solution in the propagation direction is often called Semi-Analytical Finite Element (SAFE) Method [22–24]. Similar to TLM a quadratic eigenvalue problem is solved but the approach is extended to three-dimensional domains. The cross-section is described by classical Finite Elements. Model decomposition can be applied to propagate a given signal along a section of the waveguide [25,26]. Even coupling to full FEM models is possible and allows the inclusion of defects [27,28] and plate edges [29].

This paper is based on the Scaled Boundary Finite Element Method (SBFEM) [30–33]. The SBFEM is a general semi-analytical method discretizing the boundary of the domain only and has shown to combine several advantages of FEM and the Boundary Element Method (BEM). The original idea of this approach was to describe unbounded domains using Finite Elements [34,35] without being bound to layers such as the TLM. Numerous applications of SBFEM have arisen since its first conception and it is possible to describe bounded and unbounded domains in the time domain and frequency domain [36–40].

The SBFEM has recently been introduced to compute dispersion curves and mode shapes for infinitely long waveguides of constant cross-section [41,42]. A formulation in the frequency domain allows for rapid computation of dispersion curves and mode shapes for given frequencies. Similar to the TLM and the SAFE Method, an eigenvalue problem needs to be solved. New publications introduced novel approaches for boundary conditions in propagation direction [44,45] and most recently an innovative proposal to compute stiffness matrices based on this approach has been presented [46]. Coupling with stiffness matrices derived by FEM or coupling with stiffness matrices derived by classical SBFEM is easily possible and extends the possible range of applications radically.

While highly effective for single frequencies, it remains a frequency discrete method. Especially for signals in the time domain the number of frequencies of interest can lead to limitations in efficiency in this approach. Thus, tracing or, more precisely, mode-tracing of dispersion curves and mode shapes is a promising solution to overcome these limitations. Straightforward approaches like extrapolation or interpolation of previously computed solutions have been proposed in the context of the SBFEM [43] and FEM [52]. While assisted by inverse iteration those approaches are fast and applicable in many cases, but they can fail if mode behavior is too unpredictable. For more complex domains, modes can have strong and weak coupling effects [48,49,47], such as veering or locking. These effects lead to sudden changes in the model behavior and weaken the applicability of known solution approaches as the sudden change in behavior is not consistent with the behavior of previously computed solutions. If the sudden change of behavior is not identified, the applied extrapolation can lead to a different or even an unwanted set of solutions.

This paper proposes a different approach which uses analytically computed derivatives of the underlying eigenvalue problem [50,51]. Information about coupling effects is easily found in the eigenvalue derivatives, even at a distance of the critical frequency. Instead of simply adjusting the step size, this approach will compute a high number of derivatives. Thereby, it is possible to describe large parts of the dispersion relation and mode shapes by Taylor approximation and to describe a radical change of behavior through coupling effects by Padé approximation.

To formulate and discuss this approach, three steps are described in this paper. Firstly, the SBFEM formulation for elastic guided waves is presented and the resulting eigenvalue problem is shown. Secondly, the process of differentiating an eigenvalue problem is discussed and the approximations with Taylor and Padé are introduced. Lastly, the approximation process is applied to numerical examples. Especially the behavior of coupled modes at critical frequencies is discussed.

2. Theory

This section presents an overview of mode-tracing with high order derivatives in the context of the SBFEM. First of all, the governing equations of elastic guided waves are briefly introduced and the major steps of SBFEM formulations are shown. Next, the resulting eigenvalue problem is differentiated and problem specific optimizations are proposed. Lastly, the approximation process with higher derivatives of eigenvalue problems is described.

2.1. Governing equations

For a body in vacuum the relation between the strain $\boldsymbol{\sigma}$ and the displacements $\mathbf{u}$ is described by

$$\mathbf{L}^{\mathbf{T}}\boldsymbol{\sigma} + \omega^2\rho\mathbf{u} = 0 \tag{1}$$

as the governing set of equations for three-dimensional linear elastodynamics, if all body forces vanish [33]. $\mathbf{L}$ is a differential operator and the relation given by (1) is dependent on the angular frequency ω and the mass density ρ. By introducing the elasticity matrix $\mathbf{D}$ and by applying Hooke's law the relation between $\boldsymbol{\sigma}$ and the strain $\boldsymbol{\varepsilon}$ reads

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}. \tag{2}$$

Eq. (2) is transformed into

$$\boldsymbol{\sigma} = \mathbf{DLu}. \tag{3}$$

by describing strain $\boldsymbol{\varepsilon} = \mathbf{Lu}$ through the displacement amplitudes $\mathbf{u}$.

Previous publications [41–43] have derived the theoretical background to compute dispersion relations of guided waves in plates, axisymmetric structures and general three-dimensional waveguides using SBFEM. For the novel mode-tracing approach two aspects are essential. Firstly, the cross section of all mentioned geometries is discretized by means of Finite Elements while an analytical ansatz describes the direction of wave propagation. Secondly, the displacement field $\mathbf{u}$ or, more precisely, the values scaling the Finite Element shape functions $\hat{\mathbf{u}}$ are computed by solving an eigenvalue problem