Ultrasonics 66 (2016) 27–33

Contents lists available at ScienceDirect

Ultrasonics

journal homepage: www.elsevier.com/locate/ultras

Acoustic backscattering and radiation force on a rigid elliptical cylinder in plane progressive waves

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ARTICLE INFO

Article history:
Received 1 September 2015
Received in revised form 3 December 2015
Accepted 7 December 2015
Available online 15 December 2015

Keywords:
Elliptic cylinder
Scattering
Radiation force
Scattering efficiency
Modal matching method

ABSTRACT

This work proposes a formal analytical theory using the partial-wave series expansion (PWSE) method in cylindrical coordinates, to calculate the acoustic backscattering form function as well as the radiation force-per-length on an infinitely long elliptical (non-circular) cylinder in plane progressive waves. The major (or minor) semi-axis of the ellipse coincides with the direction of the incident waves. The scattering coefficients for the rigid elliptical cylinder are determined by imposing the Neumann boundary condition for an immovable surface and solving a resulting system of linear equations by matrix inversion. The present method, which utilizes standard cylindrical (Bessel and Hankel) wave functions, presents an advantage over the solution for the scattering that is ordinarily expressed in a basis of elliptical Mathieu functions (which are generally non-orthogonal). Furthermore, an integral equation showing the direct connection of the radiation force function with the square of the scattering form function in the far-field from the scatterer (applicable for plane waves only), is noted and discussed. An important application of this integral equation is the adequate evaluation of the radiation force function from a bistatic measurement (i.e., in the polar plane) of the far-field scattering from any 2D object of arbitrary shape. Numerical predictions are evaluated for the acoustic backscattering form function and the radiation force function, which is the radiation force per unit length, per characteristic energy density, and per unit cross-sectional surface of the ellipse, with particular emphasis on the aspect ratio a/b, where a and b are the semi-axes, as well as the dimensionless size parameter kb, without the restriction to a particular range of frequencies. The results are particularly relevant in acoustic levitation, acousto-fluidics and particle dynamics applications.

1. Introduction

Acoustic scattering and radiation force are intertwined phenomena encountered in most (if not all) applications whenever the acoustical field interacts with an object, as a result of scattering and absorption. The linear momentum transfer from the waves to the object causes a steady force [1] that may push [2–5] or pull [6,7] the object toward the source.

For optimal experimental design purposes in applications ranging from biomedical acoustics [8], underwater acoustics [9], particle dynamics [10], non-destructive imaging [11,12], acousto-fluidics [13,14], and other emerging areas, computer simulations and numerical predictions of the scattering and radiation force are constantly required.

Considerable literature on the scattering by circular cylinders exists [15–24], with further extension to the non-circular (2D elliptical) geometries based on series expansions in Mathieu functions [25–30], the (null-field) *T*-matrix [31,32], the Green's function approach [33], the Finite Element Method [34], the finite difference time-domain approach [35], the Fourier Matching Method [36] based on conformal mapping [37,38], and the boundary integral equations [39]. Other formalisms have been based on the partial-wave series expansion (PWSE) decomposition using cylindrical basis wave-functions [40–42] and group theory [43], known also as the method of modal expansions [44].

Regarding the acoustic radiation force, substantial investigations have considered the infinitely-long cylindrical geometry [45–71] based on the integration of the radiation stress tensor, which involves quadratic quantities, such as the acoustical kinetic and potential energy densities and Reynolds' stress. Such quantities contain physical observables related to the mutual interaction of the incident and scattered waves, thus, demonstrating the close connection with the scattering phenomenon.

An earlier investigation [72] confirming the lack of analytical solutions for the radiation force on a cylinder when its form deviates from the circular shape, considered the case of an elliptic cylinder with infinite length by means of infinite series involving elliptical (modified) Mathieu functions. However, the generation

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http://dx.doi.org/10.1016/j.ultras.2015.12.003
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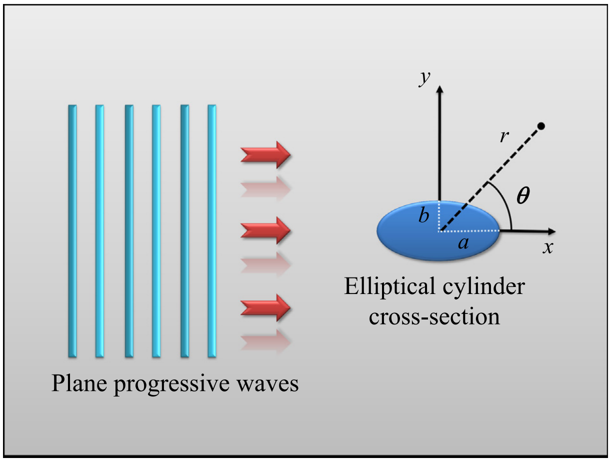

Fig. 1. The schematic describing the interaction of acoustical plane progressive waves with a rigid cylinder of elliptical cross-section in a non-viscous fluid. The semi-axes of the ellipse are denoted by a and b, respectively. The cylindrical coordinate system (r, θ, z) is referenced to the center of the elliptical cylinder, where z-axis is perpendicular to the plane of the figure.

and numerical development of the elliptical functions may not be a straightforward task, since the angular wave functions are generally non-orthogonal. This apparent complication may be alleviated by expanding the angular Mathieu functions in terms of transcendental functions. However, this algebraic manipulation requires subsequent integration before obtaining a linear set of independent equations to solve for the scattering coefficients. This algebraic manipulation may lead to spurious results if a criterion of convergence is not met, or possible numerical instabilities for the elliptical functions are not properly resolved (see for instance the second column of Fig. 2 in [30], where the large peaks seem to be unphysical). Therefore, it is of some importance to develop an improved formalism applicable to 2D objects with arbitrary cross-section using standard cylindrical-wave functions.

The aim of this work is to develop a formal analytical solution to solve for the scattering off an elliptical cylinder as well as the acoustic radiation force in plane progressive waves, where the major (or minor) semi-axis coincides with the axis of wave propagation (defined as the on-axis configuration), as shown in Fig. 1. The formulation is applicable to any range of frequencies such that either the long- or short-wavelength with respect to the size of the object can be examined. The present analytical formalism is based on the partial-wave series expansion (PWSE) method in cylindrical coordinates and the description of the total field that is an exact solution of the Helmholtz equation. The cases of rigid (fixed) ellipses with variable aspect ratios a/b are considered, and the Neumann boundary condition for an immovable surface is imposed such that a system of linear equations allows resolving for the scattering coefficients of the elliptical cylinder by matrix inversion. Subsequently, a far-field analysis of the scattering is utilized to evaluate the acoustic radiation force. The procedure for evaluating the acoustic radiation force stemming from an analysis of the far-field scattering off the object does not induce any approximation as long as non-viscous fluids are considered. This is an early result recognized by various authors [73–78] in the derivation of the acoustic radiation force components.

2. Scattering of plane progressive waves by a rigid elliptical cylinder in the on-axis configuration

Consider a plane progressive wave propagating in a nonviscous fluid, and incident upon an elliptical cylinder where one of the semi-axes lies on the axis of wave propagation x of the incident waves (i.e., on-axis configuration), as shown in Fig. 1.

The incident field can be expressed in terms of a PWSE in a system of cylindrical coordinates (r, θ, z) with its origin chosen at the center of the ellipse, as,

$$\Phi_{inc} = \phi_0 e^{-i\omega t} \sum_n i^n J_n(kr) e^{in\theta}, \tag{1}$$

where ϕ_0 is the amplitude, the summation $\sum_n = \sum_{n=-\infty}^{+\infty}$, $J_n(\cdot)$ is the cylindrical Bessel function of first kind of order n, θ is the polar angle, and k is the wave number of the incident radiation.

The scattered velocity potential field off the ellipse is also expressed as a PWSE as,

$$\Phi_{sca} = \phi_0 e^{-i\omega t} \sum_n i^n C_n H_n^{(1)}(kr) e^{in\theta}, \tag{2}$$

where C_n are the scattering coefficient to be determined by applying the Neumann boundary condition for a rigid immovable surface with the assumption that the fluid surrounding the target is nonviscous, and $H_n^{(1)}(\cdot)$ is the cylindrical Hankel function of the first kind, describing the propagation of outgoing waves in the surrounding fluid.

In 2D, the surface shape function A_θ of the ellipse depends on the polar angle θ. The equation describing the elliptical surface shape function is given by,

$$A_\theta = \left[(\cos\theta/a)^2 + (\sin\theta/b)^2 \right]^{-1/2}, \tag{3}$$

where a and b are the semi-axes of the ellipse.

The essence of the method is to apply the Neumann boundary condition [19] for the total (incident + scattered) steady-state velocity potential field for a rigid immovable ellipse at $r = A_\theta$, such that,

$$\nabla(\Phi_{inc} + \Phi_{sca}) \cdot \mathbf{n}|_{r=A_\theta} = 0, \tag{4}$$

where the normal vector $\mathbf{n}$ is expressed as,

$$\mathbf{n} = \mathbf{e_r} - \left(\frac{1}{A_\theta}\right) \frac{dA_\theta}{d\theta} \mathbf{e}_\theta, \tag{5}$$

with $\mathbf{e_r}$ and $\mathbf{e}_\theta$ denoting the outward unit vectors along the radial and polar directions, respectively.

Substituting Eqs. (1) and (2) into Eqs. (4) using (5) leads to a system of linear equations,

$$\sum_n i^n [\Gamma_n(\theta) + C_n \Upsilon_n(\theta)] = 0, \tag{6}$$

where the structural functions $\Gamma_n(\theta)$ and $\Upsilon_n(\theta)$ are expressed, respectively, as

$$\left\{ \begin{array}{c} \Gamma_n(\theta) \\ \Upsilon_n(\theta) \end{array} \right\} = e^{in\theta} \left[\begin{array}{l} k \left\{ \begin{array}{c} J_n'(kA_\theta) \\ H_n^{(1)\prime}(kA_\theta) \end{array} \right\} \\ -i\left(\frac{n}{A_\theta^2}\right) \frac{dA_\theta}{d\theta} \left\{ \begin{array}{c} J_n(kA_\theta) \\ H_n^{(1)}(kA_\theta) \end{array} \right\} \end{array} \right], \tag{7}$$

and the primes denote a derivative with respect to the argument.

Note that for a *circular* cylinder (i.e., A_θ = constant), the structural functions are independent of the polar angle θ, and thus, the scattering coefficients for the rigid circular cylinder are recovered [19] as $C_n = -J_n'(ka)/H_n^{(1)'}(ka)$. In the case of an elliptical cylinder, however, the angular dependency should be eliminated in order to solve the system of linear equations for each partial-wave n mode. Therefore, Eq. (6) is equated to a Fourier series as,

$$\sum_n i^n [\Gamma_n(\theta) + C_n \Upsilon_n(\theta)] = \sum_n [\psi_n + C_n \Omega_n] e^{in\theta} = 0. \tag{8}$$