Ultrasonics 64 (2016) 170–176

Contents lists available at ScienceDirect

Ultrasonics

journal homepage: www.elsevier.com/locate/ultras

Estimation of shear modulus in media with power law characteristics

Wei Zhang *, Sverre Holm

Department of Informatics, University of Oslo, P.O. Box 1080, NO-0316 Oslo, Norway

ARTICLE INFO

Article history:
Received 25 July 2014
Received in revised form 14 April 2015
Accepted 2 September 2015
Available online 9 September 2015

Keywords:
Fractional derivative
Shear wave
Attenuation

ABSTRACT

Shear wave propagation in tissue generated by the radiation force is usually modeled by either a lossless or a classical viscoelastic equation. However, experimental data shows power law behavior which is not consistent with those approaches. It is well known that fractional derivatives results in power laws, therefore a time fractional wave equation, the Caputo equation, which can be derived from the fractional Kelvin–Voigt stress and strain relation is tested. This equation is solved using the finite difference method with experimental parameters obtained from the existing literature. The equation is characterized by a fractional order which is also the power law exponent of the frequency dependent shear modulus. It is shown that for fractional order between 0 and 1, the equation gives smaller shear modulus than the classical model. The opposite situation applies for fractional order greater than 1. The numerical simulation also shows that the shear wave velocity method is only reliable for small losses. In our case, this is only for a small fractional order. Based on the published values of fractional order from other studies, there is therefore a chance for biased estimation of the shear modulus.

1. Introduction

Shear wave elasticity imaging (SWEI) has become an important tool for characterizing tissue properties. It is done by inducing a shear wave through the radiation force of a focused ultrasonic beam [1]. Understanding shear wave propagation and the material response of human tissue can provide more accurate information for medical diagnosis, device design and treatment planning. One key aspect of this is to find equations that accurately model wave attenuation and dispersion.

However, there are some challenges for finding accurate equations for dynamic processes that occur in biological tissues. For example, it is well known that attenuation of both shear and compressional waves follows a power law [2–4]

$$a_k(\omega) = a_0\omega^y, \tag{1}$$

where the subscript k denotes that $\alpha_k(\omega)$ is the imaginary part of the wave number k, α_0 is the absorption coefficient, ω is the angular frequency, and y is the power law exponent. The classical integer order wave equations can only model attenuation with $y = 0, 2$. Therefore attenuation with $y \neq 0, 2$ is called anomalous attenuation [5].

In some cases, the stress–strain relationship is nonlinear [6]. These studies require large relative tissue deformations in the order of 20%. Here we assume very small tissue deformations so that the medium's response is linear.

There is experimental data that shows the value of the power law y for shear wave attenuation. Holm and Sinkus analyzed data published by Asbach et al. and found $y \approx 0.73$ for shear wave propagation in liver using MR elastography [3,7]. Data for malignant lesions in breast shows $y \approx 0.13$ [8]. In cell rheology, Fabry et al. [9] suggested $y \approx 1.2$ in the frequency range from 0.01 to 10 Hz. Thus there are plenty of interesting experimental results in the literature, and this paper therefore does not report any new such results.

These values of the power law are well suited for modeling with fractional calculus. It is a powerful tool for describing such attenuation in a parsimonious way, i.e. with only a few parameters. Up to now, many researchers have used fractional wave equations to model mainly compressional waves in medical ultrasound. For instance, Treeby and Cox [10] modeled power law absorption and dispersion for propagation in photoacoustic imaging using the fractional Laplacian operator. Caputo et al. [11] simulated waves in a biological medium based on the Kelvin–Voigt fractional derivative stress–strain relation. Chen et al. [12] modeled compressional wave propagation in breast using the modified Szabo's equation which has a positive fractional derivative lossy operator.

Compared with the case for compressional waves, not so many have tried to model shear waves in elastography with arbitrary power laws. But Klatt et al. [13] fitted the fractional Zener model to data for brain and liver viscoelasticity. A unifying wave equation

* Corresponding author.
E-mail addresses: 71574252@qq.com (W. Zhang), sverre@ifi.uio.no (S. Holm).

http://dx.doi.org/10.1016/j.ultras.2015.09.003
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for shear and compressional was discussed by Holm and Sinkus [3]. Likewise, Holm and Näsholm [4] compared several fractional wave equations and discussed the conditions under which equations were suitable for compressional waves in medical ultrasound and for shear elastography respectively.

The usual way of simulating shear waves in elastography is either to assume a lossless medium [14] or to assume a viscous medium [15]. However this does not capture some of the frequency dependency of published measurement over the last decades. The purpose of this paper is therefore to build on published data on frequency dependency, and use those values in the fractional wave equations in order to study shear wave propagation and find if a variation in the power law characteristics can lead to a bias in the estimation of e.g. the shear modulus.

In this paper, we first introduce the Caputo equation which has previously been found to be a useful model in elastography [4], and analyze the dispersion relation. Then we give a brief introduction to shear wave elasticity imaging and find representative values for the power law parameter that different studies have come up with. In the following part, we present the numerical scheme for simulating fractional wave equations, and perform a simulation study using representative values for the fractional order. We then simulate both the peak displacement method and the time-to-peak method and analyze the results.

2. Lossy wave equations

A lossy wave equation can be written as

$$\nabla^2 u - \frac{1}{c_t^2}\frac{\partial^2 u}{\partial t^2} + Lu = 0, \tag{2}$$

where c_t is the propagation velocity for the transverse wave, i.e. the shear wave, u is the displacement, and L is the loss operator. Different loss operators influence attenuation and dispersion properties, and some of them are also only valid over a restricted frequency range (usually only low frequencies). For example, $Lu = \partial u/\partial t$ gives the power law exponent $y = 0$. Another case is the Stokes wave equation, which has $Lu = \partial(\nabla^2 u)/\partial t$ and $y = 2$ for low frequencies.

2.1. Fractional wave equation

In order to model power law behavior, we build on the fact that the Fourier transform of the fractional derivative of a function, $f(t)$, generalizes the result for an integer order derivative and is $(i\omega)^\alpha F(\omega)$, where $F(\omega)$ is the Fourier transform of $f(t)$. The fractional derivative is introduced via a loss operator which can be found by starting with a constitutive equation with a fractional relation between stress T and strain S [3]

$$T = \mu S + \eta\frac{\partial^\alpha S}{\partial t^\alpha}, \tag{3}$$

where μ is the stiffness, η is the viscosity, and α is the fractional order. The range for α is between 0 and 2, but mostly it is less than 1. The classical Kelvin–Voigt model is a special case of Eq. (3) if we let $\alpha = 1$. The mechanical representation of the fractional Kelvin–Voigt model is shown in Fig. 1, where the spring represents the elastic part, parameterized by the stiffness μ, and the dashpot (fractional damper) is parameterized by the viscosity η and the fractional order α. This particular mechanical model and its generalization in the fractional Zener model have turned out to be surprisingly accurate for a wide range of materials [16].

There are several ways to define the fractional derivative. Here we use the approach of Caputo because it has physically interpretable initial conditions. Other definitions can be found in [17]. The Caputo fractional derivative is

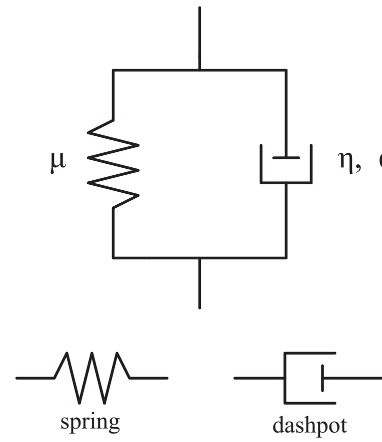

Fig. 1. The mechanical representation of the fractional Kelvin–Voigt model consisting of a Hookean spring in parallel with a fractional derivative dashpot.

$$\frac{\partial^\alpha u(t)}{\partial t^\alpha} = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-\xi)^{n-\alpha-1}\frac{d^n u(\xi)}{d\xi^n}d\xi, \qquad n-1 < \alpha < n. \tag{4}$$

Fractional calculus, allowing integrals and derivatives of any real order, is the generalization of classical calculus [17]. Fractional calculus deals with integro-differential operators and equations where the integrals are convolutions with power law type kernels. An important feature of fractional differential operator is its non-locality. In the context of Eq. (3), this means that it describes a damper with a power law memory in time.

To derive the fractional equation, we also need the relation between strain, S, and displacement, u [2]

$$S = \frac{\partial u}{\partial x}, \tag{5}$$

and the principle of conservation of momentum

$$\rho\frac{\partial^2 u}{\partial t^2} = \frac{\partial T}{\partial x}, \tag{6}$$

where ρ is the density. Combining Eqs. (3), (5) and (6), and extrapolating from 1-D to 3-D, we get

$$\nabla^2 u - \frac{1}{c_t^2}\frac{\partial^2 u}{\partial t^2} + \tau^\alpha\frac{\partial^\alpha}{\partial t^\alpha}\nabla^2 u = 0, \tag{7}$$

where the characteristic relaxation time of the medium is given by $\tau^\alpha = \eta/\mu$, and the shear wave speed is related to the elasticity by

$$\mu = \rho c_t^2. \tag{8}$$

In a lossless medium c_t is the phase velocity which is constant with frequency, while in a lossy medium, c_t is the asymptotic value of the phase velocity at zero frequency. For the convenience of the following numerical simulation, we rewrite Eq. (7) by multiplying with c_t^2

$$\frac{\partial^2 u}{\partial t^2} - c_t^2\nabla^2 u - \frac{\eta}{\rho}\frac{\partial^\alpha}{\partial t^\alpha}(\nabla^2 u) = 0, \tag{9}$$

where $\eta/\rho = c_t^2\tau^\alpha$.

It should be noted Eq. (7) was first derived by Caputo in [18] and therefore we call it the *Caputo wave equation* here. Holm and Sinkus [3] showed that Eq. (7) can be used both for shear waves and compressional waves. This was further developed in [4] where it was shown that it is a simplified version of a fractional Zener wave equation. The Caputo wave equation is valid for reasonable values of the product $\omega\tau > 1$ as long as it is not too large. This is satisfied here where the maximum value of the product will turn out to be 28.7 (Section 3). This paper is restricted to linear models