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Extended optical theorem for scalar monochromatic acoustical beams of arbitrary wavefront in cylindrical coordinates

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ABSTRACT

One of the fundamental theorems in (optical, acoustical, quantum, gravitational) wave scattering is the optical theorem for plane waves, which relates the extinction cross-section to the forward scattering complex amplitude function. In this analysis, the optical theorem is extended for the case of 3D-beams of arbitrary character in a cylindrical coordinates system for any angle of incidence and any scattering angle. Generalized analytical expressions for the extinction, absorption, scattering cross-sections and efficiency factors are derived in the framework of the scalar resonance scattering theory for an object of arbitrary shape. The analysis reveals the presence of an interference scattering cross-section term, which describes interference between the diffracted or specularly reflected inelastic (Franz) waves with the resonance elastic waves. Moreover, an alternate expression for the extinction cross-section, which relates the resonance cross-section with the scattering cross-section for an impenetrable object, is obtained, suggesting an improved method for particle characterization. Cross-section expressions are also derived for known acoustical wavefronts centered on the object, defined as the on-axis case. The extended optical theorem in cylindrical coordinates can be applied to evaluate the extinction efficiency from any object of arbitrary geometry placed on or off the axis of the incident beam. Applications in acoustics, optics, and quantum mechanics should benefit from this analysis in the context of wave scattering theory and other phenomena closely connected to it, such as the multiple scattering by many particles, as well as the radiation force and torque.

1. Introduction

Scattering and absorption are the two most significant mechanisms that affect wave propagation throughout a medium. Understanding the physical nature of these effects during wave transport allows adequate characterization of the medium of wave propagation by deconvolution of scattering and attenuation spectra.

The most rigorous description of these phenomena is interpreted by means of the so-called "optical theorem" (OT) [1–8], which constitutes a general law of wave scattering theory. It is otherwise known as the "extinction theorem" [7], which relates the extinction cross section of an object of arbitrary geometry placed in the field of monochromatic plane waves to its forward scattering amplitude, which is the scattered wave amplitude measured in the far field along the forward direction of wave propagation (p. 20 in [9]).

An analogous OT has been derived for the acoustical scattering by baffled membranes and plates in the direction of the specularly reflected waves [10], nonetheless, in scalar wave scattering applications, there has been a need to account for the plane waves scattered at any angle, not only restricted to the forward or backwards directions. An integral relation for the angle-dependent scattering amplitude of plane waves has been therefore derived from the standpoint of quantum theory [11–13], which constitutes a "generalized optical theorem" (GOT). Subsequently, it has been applied in the context of electron diffraction theory [14], scalar optical [15] and evanescent waves [16], and acoustic backscattering by elastic targets (with no internal dissipation) having inversion symmetry [17]. Additional extensions for the GOT to study vectorial wave phenomena related to electromagnetic waves [18,19], elastic waves [20], surface waves [21] and layered media [22] have been formulated as well.

The relevance of the GOT relies in the quantitative and accurate evaluation of cross sections rather than numerical integration procedures [8]. In essence, the extinction, scattering, and absorption cross sections (or power) are meaningful measures of the object scattering and absorption properties. As such, most practical applications and experimental methods (with some specific applications discussed in [23]), tools and devices involving wave scattering phenomena (such as medical, non-destructive, sonar

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and underwater imaging, multiphase flow characterization, spectroscopy and particle sizing in dilute suspensions, near-field diffraction X-ray and ultrasonic tomography, microscopy, and inverse scattering, to name a few) would benefit from an *extended* GOT formalism, which accounts for the character of the incident wavefront, as opposed to plane (unbounded) waves. For convenience, it is termed; extended optical theorem (EOT).

An EOT has been formulated for quantum beams [24], optical Gaussian beams [25] (and others of arbitrary shape [26]) that possess some degree of amplitude roll-off in the transverse direction, since the classical OT is no longer valid for shaped beams. Along that line of research, an acoustical EOT has been established for any scalar beam (not only limited to non-diffracting beams [27]) of arbitrary character incident upon an object of arbitrary shape, and for any incidence and scattering angles [23], based on the partial-wave series expansion (PWSE) method in spherical coordinates. Though the formalism developed in Ref. [23] is applicable to any object of arbitrary shape, computing the extinction, scattering, and absorption cross sections (or their corresponding efficiencies) of elongated cylindrical-like objects with the EOT in spherical coordinates, may lead to numerical instabilities and inaccuracies [28]. These challenges arise from taking a large number of spherical partial-waves to fit a non-spherical cylindrical-like object [29] in order to ensure proper convergence of the series. Nevertheless, these complications should not be interpreted as a lack of rigor of the PWSE formalism [23], rather the method in spherical coordinates may not be entirely suitable from a computational standpoint for the analysis of cylindrical-like objects with arbitrary shape. Therefore, it is of some importance to develop an analytical formalism suitable for elongated objects in cylindrical coordinates.

Here, a generalized formulation in cylindrical coordinates applicable to any acoustical beam of arbitrary character in 3D is developed for a scatterer of arbitrary geometry and size, located arbitrarily in space. Note that a related work for *optical/electromagnetic* beams has been recently considered [30]. The EOT in cylindrical coordinates provides generalized PWSEs for the extinction, absorption, and scattering cross sections in terms of the non-dimensional beam-shape and scattering coefficients of the object. Furthermore, resonance, background and interference cross-sections are defined in the framework of the generalized resonance scattering theory for arbitrary beams, and their corresponding analytical expressions are derived. Efficiency factors are also computed for a viscoelastic cylinder centered on the axis of a zeroth-order quasi-Gaussian beam, chosen as an example to illustrate the theory.

2. Theoretical analysis

Consider a scalar acoustical beam of angular frequency ω incident along an arbitrary direction on a viscoelastic object of arbitrary geometry immersed in a non-viscous fluid (Fig. 1). A time-dependence in the form of $e^{-i\omega t}$ is assumed, but omitted from the equations for convenience.

For a 3D arbitrary-shaped beam composed of monochromatic waves, the incident pressure field in the frequency domain can be expressed in a cylindrical coordinates system as [31],

$$p_i(r,\theta,z,\omega) = p_0 \sum_{n=-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} b_n(k_z) J_n(k_r r) e^{ik_z z} dk_z \right] e^{in\theta}, \quad (1)$$

where p_0 is the pressure amplitude in the absence of the object, $J_n(\cdot)$ is the cylindrical Bessel function of the first kind, k_r and k_z are the radial and axial wave-numbers, respectively, defined as $k^2 = k_r^2 + k_z^2$ [31], where k is the wavenumber of the incident radiation (for a 2D beam, k_z is a discrete (real) number [32]). The parameters r and θ are the radial distance and polar angle in the (x, y) plane, respectively. In the generalized case of an object of arbitrary shape in 3D, the radial distance $r = r(\theta, z)$. The factors $b_n(k_z)$ correspond to the beam-shape coefficients (BSCs) that solely describe the incident beam of arbitrary wavefront. They are determined as follows; first, the relationship of the two-dimensional Fourier transform (in θ and z) of the incident pressure field $\mathscr{F}\{p_i\}$ is derived. Then, its inverse function $\mathscr{F}^{-1}\{\mathscr{F}\{p_i\}\}$ is expressed in terms of a Fourier series and a transform (p. 125 in [31]), and the result is equated with Eq. (1), leading to the expression,

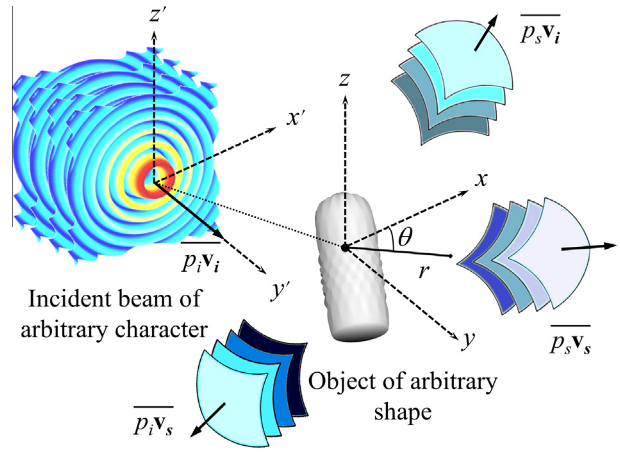

Fig. 1. An elongated object of arbitrary shape placed in the field of an incident acoustical beam of arbitrary wavefront in cylindrical coordinates (r, θ, z). The primed coordinate system has its origin at the center of the beam, while the unprimed coordinate system is referenced to the object.

$$b_n(k_z) = \frac{1}{(2\pi)^2 p_0 J_n(k_r r)} \int_0^{2\pi} \left[\int_{-\infty}^{+\infty} p_i(r,\theta,z,\omega) e^{-ik_z z} dz \right] e^{-in\theta} d\theta. \quad (2)$$

If a given beam expression for the incident pressure is an exact solution of the Hemlholtz's equation, the r dependence in Eq. (2) cancels out, and the BSCs are functions of k_z and n alone.

The scattered wave can be represented by a scattered pressure field expressed as,

$$p_s(r,\theta,z,\omega) = p_0 \sum_{n=-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} b_n(k_z) C_n H_n^{(1)}(k_r r) e^{ik_z z} dk_z \right] e^{in\theta}, \quad (3)$$

where $H_n^{(1)}(\cdot)$ is the cylindrical Hankel function of the first kind of order n, and C_n are the non-dimensional scattering coefficients that depend on the mechanical properties of the object of arbitrary shape and the surrounding nonviscous fluid. When the object takes the form of a circular cylinder, the radial distance r is constant (independent of θ and z), and the scattering coefficients C_n can be exactly described in terms of known cylindrical Bessel, Neumann and Hankel functions and their derivatives, as well as the mechanical properties of the cylindrical material [33].

It proves convenient to use the far-field expressions for the incident and scattered acoustic fields in a nonviscous fluid to evaluate the cross-sections, by using the asymptotic limits for the cylindrical Bessel and Hankel functions with large arguments. Thus, in the far-field limit, Eqs. (1) and (3) become

$$p_i(r,\theta,z,\omega) \underset{k_r r \to \infty}{\approx} p_0 \sum_{n=-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} \sqrt{\frac{2}{\pi k_r r}} b_n \cos\left(k_r r - \frac{n\pi}{2} - \frac{\pi}{4}\right) e^{ik_z z} dk_z \right] e^{in\theta}, \quad (4)$$

$$p_s(r,\theta,z,\omega) \underset{k_r r \to \infty}{\approx} p_0 \sum_{n=-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} \sqrt{\frac{2}{i\pi k_r r}} b_n C_n e^{i(k_z z + k_r r)} dk_z \right] i^{-n} e^{in\theta}. \quad (5)$$