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Reciprocity principle for scattered fields from discontinuities in waveguides

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ABSTRACT

This study investigates the scattering of guided waves from a discontinuity exploiting the principle of reciprocity in elastodynamics, written in a form that applies to waveguides. The coefficients of reflection and transmission for an arbitrary mode can be derived as long as the principle of reciprocity is satisfied at the discontinuity. Two elastodynamic states are related by the reciprocity. One is the response of the waveguide in the presence of the discontinuity, with the scattered fields expressed as a superposition of wave modes. The other state is the response of the waveguide in the absence of the discontinuity oscillating according to an arbitrary mode. The semi-analytical finite element method is applied to derive the needed dispersion relation and wave mode shapes. An application to a solid cylinder with a symmetric double change of cross-section is presented. This model is assumed to be representative of a damaged rod. The coefficients of reflection and transmission of longitudinal waves are investigated for selected values of notch length and varying depth.

1. Introduction

Guided waves have played an important role in nondestructive testing. Applications range from the detection of corrosion in pipes or cracks in weldments to the evaluation of the state of prestress [1]. Compared to conventional ultrasonic methods, the advantage of guided waves is offering minimal attenuation, thus providing larger inspection ranges. Moreover, if compared to modal analysis [2,3], guided waves are more sensitive because of the use of higher frequencies. Practical guided-wave ultrasonic testing is done by sending a signal along a waveguide and interpreting the scattered response. In the presence of a defect, this response consists of a complex superposition of waves. The characteristics of a discontinuity must be related to the amplitude of scattered fields to obtain a quantitative evaluation of the discontinuity itself, and hence solve the inverse problem [4].

In the simplest case, one single mode signal is used. This can be converted to a multimode reflected or transmitted signal when interacting with defects and discontinuities. Calculating the related scattering coefficients is challenging. Many researchers investigated the problem, and a complete review of the vast literature is beyond the scope of the present paper. Shear and Lamb waves in plates have attracted the most attention, while relatively little research deals with rods, the focus of this study. Different approaches are available, but, in general, it is necessary to resort to numerical methods.

To clearly detect a notch, signals must be used with a wavelength of the same order of magnitude as the notch extension. Hence, if minor defects need to be detected, small wavelengths and high frequencies are necessary [4–6]. In this case, the hypothesis of all beam theories, namely that the cross-section of the waveguide remains undeformed, no longer applies [7]. This calls for different approaches for the calculus of the scattering coefficients, which can be classified into: methods based on wave expansions, often referred to as mode-matching or modal decomposition methods [8–16], finite element methods [17–19], or hybrid numerical methods combining finite element formulations with wave or boundary element approaches [6,20–24]. In these hybrid methods only the region where the defect is present is modeled with finite elements. Equilibrium and continuity equations are then enforced at the boundaries. In modal decomposition methods, the scattering coefficients deriving from the enforcement of equilibrium and continuity equations can be determined by resorting to least square methods or to variational principles. The benefit of variational principles as a tool for solution, such as the procedure presented here, is that they require fewer wave modes in the modal expansion than least square techniques [10,11].

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This study uses a method whereby the reflection and transmission coefficients in waveguides are obtained, using the principle of reciprocity in elastodynamics [25,26], whose effectiveness is to date underestimated. Reciprocity theorems in elasticity theory provide relations between the elastic solutions (stresses, displacements and strains) of two different loading states (body and surface forces, constraints). The idea of connecting two states with reciprocity to determine scattering coefficients was developed in [9]. Analogously, here, a *virtual* or *test* wave, whose solution is known, is used to obtain information on another elastodynamic state, that is the response of the waveguide in the presence of the discontinuity. Differently from [9], we assume that the defect has a finite extension which sustains wave modes. The method can be included in the class of modal decomposition methods whose scattering coefficients are determined based on energy principles, and therefore has similar advantages (reduced computational time when compared to FE or BE, reduced number of modes to be included in the response when compared to solutions obtained with least square methods [11]) and disadvantages (difficult treatment of defects of complex shape).

For the description of the wave propagation in the waveguide, a technique based on the factorization of the function describing the displacement field is used, as shown in [27]. According to this technique, the displacement field is represented as a product of a field defined over the cross-section, which is discretized, and of a complex exponential which prescribes the propagation along the waveguide axis (Semi Analytical Finite Element method, SAFE). Given a frequency, the solution of the eigenvalue problem, derived from the equilibrium equations, provides the dispersion relation and wave mode shapes, which are used to express the waveguide response in terms of mode superposition. This SAFE approach reduces the FEM computational burden, and waveguides of complex cross-section can be easily dealt with [28].

Exploiting the reciprocity relationships between wave fields, reflection and transmission coefficients of a wave encountering a discontinuity can be determined by writing a squared system of algebraic equations requiring that the reciprocity is satisfied at the discontinuity itself. This also avoids having to solve equilibrium and continuity equations in a least square sense [13,24]. The proposed method can easily be applied to vertical discontinuities, be they symmetric or antisymmetric, and to arbitrary cross sections. It could also be extended to irregular changes of cross-section or defects of complex shape resorting to an approximation of these changes by a sequence of stair-steps, as in [15].

The study addresses the nature of scattering and has general applicability. Once the reflection and transmission coefficients are known, the response can be modeled again by wave-mode superposition. An application is presented regarding a rod with a sharp change of transverse section by varying axial extent, depth of the notch and frequency of the incident wave.

2. Guided waves in a rod

The equations of free vibrations for a three-dimensional elastic homogeneous isotropic solid with a free-stress boundary are:

$$\sigma_{ij,j} = \rho \ddot{u}_i \quad \text{in} \quad S, \quad \sigma_{ij} n_j = 0 \quad \text{on} \quad \partial S \tag{1}$$

with $i,j = 1,2,3$, where ρ is the material density and $\sigma_{ij} = \sigma_{ji}$ are the components of the symmetric stress tensor. The constitutive relationships are: $\sigma_{ij} = C_{ijkl}\epsilon_{kl}$, with $C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$, where δ_{ij} is the Kronecker delta, λ and μ are the Lamé coefficients. C_{ijkl} are the components of the stiffness tensor, for which the symmetries $C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}$ hold, and ϵ_{kl} are the components of the infinitesimal strain tensor, which can be written, in terms of displacement variables u_i, as: $\epsilon_{kl} = (u_{k,l} + u_{l,k})/2$. In such a way, an explicit formulation of the elastic problem in terms of u_i is obtained.

Let us consider the system depicted in Fig. 1, consisting of an infinite solid cylinder. The coordinate system attached to its axis is x_1, x_2, x_3, where x_1 and x_2 are the cross-section coordinates and x_3 is the axial coordinate. For such a geometry, we look for solutions of the kind

$$u_i(x_1, x_2, x_3, t) = U_i(x_1, x_2) e^{\mathrm{i}(kx_3 - \omega t)} \tag{2}$$

which represent harmonic guided waves (wave modes) propagating along x_3 with frequency ω and wave-number k, i being the imaginary unit. $U_i(x_1, x_2)$ is the i-th component of the related wave mode shape. Requiring that the displacement field has an expression such as (2) all the derivatives with respect to x_3 from the equations of motion can be removed since:

$$u_{i,3} = \mathrm{i}kU_i e^{\mathrm{i}(kx_3 - \omega t)} \quad u_{i,33} = -k^2 U_i e^{\mathrm{i}(kx_3 - \omega t)}, \tag{3}$$

where the dependence of $u_{i,3}$ and $u_{i,33}$ on x_1, x_2, x_3 has been omitted for the sake of brevity. Thus, the initial three-dimensional Eq. (1) of motion are recast into a quadratic eigenvalue problem on the cross-section S of the waveguide where k^2 is the sought eigenvalue for each ω. It is convenient to reformulate the eigenvalue problem in a form linearly depending on k. This is done by adding three new variables V_1, V_2, V_3 contained in the vector $\mathbf{V} = k\mathbf{U}$. The resulting equations are:

$$\begin{aligned}
&(\lambda + 2\mu)U_{1,11} + \mu U_{1,22} + (\lambda + \mu)(\mathrm{i}kU_{3,1} + U_{2,12}) + \rho\omega^2 U_1 = k\mu V_1 \\
&\mu U_{2,11} + (\lambda + 2\mu)U_{2,22} + (\lambda + \mu)(\mathrm{i}kU_{3,2} + U_{1,12}) + \rho\omega^2 U_2 = k\mu V_2 \\
&\mu(U_{3,11} + U_{3,22}) + \mathrm{i}k(\lambda + \mu)U_{1,1} + \mathrm{i}k(\lambda + \mu)U_{2,2} + \rho\omega^2 U_3 = k(\lambda + 2\mu)V_3 \\
&V_1 = kU_1 \qquad V_2 = kU_2 \qquad V_3 = kU_3
\end{aligned} \tag{4}$$

To these field equations, the boundary conditions of free stress must also be imposed. For each given frequency, the resulting eigenvalue problem has an infinite number of eigenvalues and eigenvectors representing, respectively, wave-numbers and wave modes.

The eigenvalue problem (4) is solved here by calculating a discrete numerical solution with an approach called the semi-analytical finite element method (SAFE), as the solution is in part numerical ($U_i(x_1, x_2)$, defined on the cross section) and in part it preserves its continuous features ($e^{\mathrm{i}(x_3 k - \omega t)}$). The eigenvectors $U_i(x_1, x_2)$ are assumed to be power-normalized. Let us call Σ_{ij} the ij component of the modal tensor of stress related to a given eigenfunction, obtained substituting the related strain field in the constitutive equation. Power normalization requires that the wave modes satisfy:

$$\frac{\pi^2 \omega}{2} \int_S U_j(x_1, x_2) \Sigma_{ij}(x_1, x_2) dS = 1, \tag{5}$$

which expresses the time average of the power flow of a single mode through the cross-section.

For a more general representation, the non-dimensional angular frequency Ω and wave-number K are introduced:

$$\Omega = \omega r_0 / c_0 \quad K = kr_0 \tag{6}$$

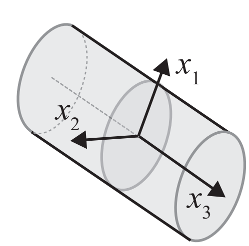

Fig. 1. Infinite solid cylinder with circular cross-section.