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Analysis of second harmonic guided waves in pipes using a large-radius asymptotic approximation for axis-symmetric longitudinal modes

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ABSTRACT

Theoretical formulation for the problem of second harmonic guided waves in pipes is presented from the principles of continuum mechanics. The formulation is carried out in the reference configuration of the pipe with an emphasis on the correct use of the "Divergence" operator in the reference configuration. Second harmonic guided wave generation from axis-symmetric longitudinal guided wave modes is studied. A large radius asymptotic approximation for the wave structures in pipe is studied and an error estimate for the same is obtained. Comparison with the corresponding modes in a plate and the analogy to second harmonic guided wave generation in plates is presented.

1. Introduction

Use of nonlinear ultrasound for monitoring evolution and characterization of microstructure has been a topic of interest for several decades. Hikata et al. [1] and Hikata and Elbaum [2,3] analyzed the problem of second and third harmonic generation due to the presence of dislocations in materials. Cantrell [4,5] used nonlinear ultrasound for characterizing fatigue damage accumulation in metals and also correlated the acoustic nonlinearity parameter (β) to the percent remaining life of the material. While the earlier work focused on using bulk-waves, guided waves are now considered as an option for generating cumulative harmonics. The works by Deng [6,7] and de Lima and Hamilton [8] laid the basis for the theoretical development of the problem of second harmonic guided wave generation in plates. Later, Srivastava and Lanza di Scalea [9] extended the problem formulation for higher harmonics in plates. Matsuda and Biwa [10] and Chillara and Lissenden [11] presented two independent approaches to arrive at the guided wave modes that are capable of generating cumulative second harmonics. Chillara and Lissenden [12] presented a generalized approach to study the interaction of guided wave modes in plates and presented an analysis to predict the nature of higher harmonics in plates. While the major portion of work on nonlinear guided waves focused on plates, de Lima and Hamilton [13] and Srivastava and Lanza di Scalea [14] extended the approach for wave guides with arbitrary but constant cross section and rods respectively.

In this article, we present a fully consistent formulation for the problem of nonlinear guided waves in pipes. The problem formulation is carried out using the principles of continuum mechanics and in the reference configuration of the pipe. This article deals with second harmonic guided wave generation from axis-symmetric longitudinal modes and we will consider the second harmonic generation from torsional or flexural modes in a subsequent article. As the guided wave modes in the pipes cannot be classified as either symmetric or antisymmetric like that in a plate, we cannot obtain generalized conclusions regarding the nature of second harmonics. For example, we have only symmetric modes generated as second harmonics in plates [9,11]. So, we introduce the notion of asymptotic modes which behave as symmetric/antisymmetric modes in an asymptotic sense. Then, we show that the conclusions obtained for second harmonic guided wave generation in plates can be appropriately extended to pipes using the asymptotic solution.

The content of this article is organized as follows. Section 2 presents the preliminaries on continuum mechanics and an accurate expression for the "Divergence" of the first Piola–Kirchhoff stress tensor to be used for the problem of second harmonic guided waves in pipes. Section 3 gives the problem formulation for second harmonic guided wave generation from axis-symmetric longitudinal modes. Then Section 4 provides the asymptotic solution that we use in this article and Section 5 presents a comparison between the plate and the asymptotic solution to draw conclusions regarding the nature of second harmonic guided waves in pipes. Finally, Section 6 presents conclusions.

2. Theory

Let $\boldsymbol{B}_{\kappa}$ denote the reference configuration and $\boldsymbol{B}$ denote the current configuration of the body. Let $\boldsymbol{\chi}$: $\boldsymbol{B}_{\kappa} \rightarrow \boldsymbol{B}$ denote the motion. Denote the position of a material particle in the reference

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configuration by $\mathbf{X}$, and that in the current configuration by $\boldsymbol{x}$. In the present context, it is convenient to use cylindrical polar coordinates to describe the motion. Use the unit vectors $(\boldsymbol{u}_\mathbf{R}, \boldsymbol{u}_\Theta, \boldsymbol{u}_\mathbf{Z})$ and the coordinates (R,Θ,Z) for the reference configuration and $(\boldsymbol{u}_\mathbf{r}, \boldsymbol{u}_\theta, \boldsymbol{u}_\mathbf{z})$ and (r,θ,z) for the current configuration. Hence,

$$B_\kappa = \{(R,\Theta,Z) : R_1 < R < R_2, 0 \leqslant \Theta < 2\pi, -\infty < Z < \infty\}$$
$$B = \{(r,\theta,z) : 0 < r < \infty, 0 \leqslant< 2\pi, -\infty < z < \infty\}$$

where R_1 and R_2 are the inner and outer radii of the pipe respectively. Thus, for a general motion $\boldsymbol{\chi}$, we have $r = r(R,\Theta,Z)$, $\theta = \theta(R,\Theta,Z)$ and $z = z(R,\Theta,Z)$.

Let $\mathbf{F}$ denote the deformation gradient, a two-point tensor mapping reference and current configuration and satisfying the following relation

$$\boldsymbol{dx} = \mathbf{F}\ \mathbf{dX}. \tag{1}$$

$\mathbf{F}$ can be represented as follows

$$\begin{aligned}\mathbf{F} = {} & \mathbf{F}_{rR}\boldsymbol{u}_\mathbf{r} \otimes \boldsymbol{u}_\mathbf{R} + \mathbf{F}_{r\Theta}\boldsymbol{u}_\mathbf{r} \otimes \boldsymbol{u}_\Theta + \mathbf{F}_{rZ}\boldsymbol{u}_\mathbf{r} \otimes \boldsymbol{u}_\mathbf{Z} + \mathbf{F}_{\theta R}\boldsymbol{u}_\theta \otimes \boldsymbol{u}_\mathbf{R} \\ & + \mathbf{F}_{\theta\Theta}\boldsymbol{u}_\theta \otimes \boldsymbol{u}_\Theta + \mathbf{F}_{\theta Z}\boldsymbol{u}_\theta \otimes \boldsymbol{u}_\mathbf{Z} + \mathbf{F}_{zR}\boldsymbol{u}_z \otimes \boldsymbol{u}_\mathbf{R} + \mathbf{F}_{z\Theta}\boldsymbol{u}_z \otimes \boldsymbol{u}_\Theta \\ & + \mathbf{F}_{zZ}\boldsymbol{u}_z \otimes \boldsymbol{u}_\mathbf{Z}\end{aligned} \tag{2}$$

where

$$\begin{aligned} & F_{rR} = \frac{\partial r}{\partial R}; \quad F_{r\Theta} = \frac{1}{R}\frac{\partial r}{\partial \Theta}; \quad F_{rZ} = \frac{\partial r}{\partial Z} \\ & F_{\theta R} = r\frac{\partial \theta}{\partial R}; \quad F_{\theta\Theta} = \frac{r}{R}\frac{\partial \theta}{\partial \Theta}; \quad F_{\theta Z} = r\frac{\partial \theta}{\partial Z} \\ & F_{zR} = \frac{\partial z}{\partial R}; \quad F_{z\Theta} = \frac{1}{R}\frac{\partial z}{\partial \Theta}; \quad F_{zZ} = \frac{\partial z}{\partial Z}. \end{aligned} \tag{3}$$

The Lagrangian strain $\mathbf{E}$ is given by

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^\mathbf{T}\mathbf{F} - \mathbf{I}). \tag{4}$$

Consider the material of the pipe to be hyperelastic with a strain energy function $\widetilde{\mathbf{W}}(\mathbf{E})$ given by (see [15]).

$$\widetilde{\mathbf{W}}(\mathbf{E}) = \frac{1}{2}\lambda(\mathbf{tr}(\mathbf{E}))^2 + \boldsymbol{\mu}\mathbf{tr}(\mathbf{E}^2) + \frac{1}{3}\mathbf{C}(\mathbf{tr}(\mathbf{E}))^3 + \mathbf{Btr}(\mathbf{E})\mathbf{tr}(\mathbf{E}^2) + \frac{1}{3}\mathbf{Atr}(\mathbf{E}^3)$$

where λ, μ are the Lame's constants and A, B and C are third order elastic constants.

The second Piola–Kirchhoff stress tensor $\mathbf{T_{RR}}$ can be obtained by using the following relation

$$\mathbf{T_{RR}} = \frac{\partial \widetilde{\mathbf{W}}(\mathbf{E})}{\partial \mathbf{E}} \tag{5}$$

which gives

$$\mathbf{T_{RR}} = \lambda\mathbf{tr}(\mathbf{E})\mathbf{I} + \mathbf{2}\boldsymbol{\mu}\mathbf{E} + \mathbf{C}(\mathbf{tr}(\mathbf{E}))^\mathbf{2}\mathbf{I} + \mathbf{Btr}(\mathbf{E}^\mathbf{2})\mathbf{I} + \mathbf{2Btr}(\mathbf{E})\mathbf{E} + \mathbf{AE}^\mathbf{2}. \tag{6}$$

The first Piola–Kirchhoff tensor can be obtained using the relation

$$\mathbf{S} = \mathbf{FT_{RR}} \tag{7}$$

Note that the first Piola–Kirchhoff stress, like the deformation gradient is a two point tensor and can be represented as

$$\begin{aligned}\mathbf{S} = {} & \mathbf{S}_{rR}\boldsymbol{u}_\mathbf{r} \otimes \boldsymbol{u}_\mathbf{R} + \mathbf{S}_{r\Theta}\boldsymbol{u}_\mathbf{r} \otimes \boldsymbol{u}_\Theta + \mathbf{S}_{rZ}\boldsymbol{u}_\mathbf{r} \otimes \boldsymbol{u}_\mathbf{Z} + \mathbf{S}_{\theta R}\boldsymbol{u}_\theta\boldsymbol{u}_\mathbf{R} + \mathbf{S}_{\theta\Theta}\boldsymbol{u}_\theta\boldsymbol{u}_\Theta \\ & + \mathbf{S}_{\theta Z}\boldsymbol{u}_\theta\boldsymbol{u}_\mathbf{Z} + \mathbf{S}_{zR}\boldsymbol{u}_z \otimes \boldsymbol{u}_\mathbf{R} + \mathbf{S}_{z\Theta}\boldsymbol{u}_z \otimes \boldsymbol{u}_\Theta + \mathbf{S}_{zZ}\boldsymbol{u}_z \otimes \boldsymbol{u}_\mathbf{Z}\end{aligned} \tag{8}$$

where S_{rR} and others are the components of $\mathbf{S}$ when represented using a mixed basis.

With an eye towards the wave equation, we now consider the expression for "Divergence" of a two point tensor $\mathbf{S}$ where the "Div" denotes divergence taken with respect to the coordinates in reference configuration. $\boldsymbol{Div}(\mathbf{S})$ is a vector in the current configuration and defined as follows

$$\boldsymbol{Div}(\mathbf{S}^\mathbf{T}\boldsymbol{a}) = \boldsymbol{Div}(\mathbf{S}) \cdot \boldsymbol{a} \quad \forall \ constant \ \ vector \ fields \ \boldsymbol{a} \in \boldsymbol{B} \tag{9}$$

The divergence of the first Piola–Kirchhoff tensor is given by the following expression (see Appendix):

$$\begin{aligned}\boldsymbol{Div}(\mathbf{S}) = {} & \left\{\frac{\partial \boldsymbol{S}_{rR}}{\partial \boldsymbol{R}} + \frac{\boldsymbol{S}_{rR}}{\boldsymbol{R}} + \frac{\mathbf{1}}{\boldsymbol{R}}\frac{\partial \boldsymbol{S}_{r\Theta}}{\partial \Theta} + \frac{\partial \boldsymbol{S}_{rZ}}{\partial \boldsymbol{Z}} - \boldsymbol{S}_{\theta R}\frac{\partial \theta}{\partial \boldsymbol{R}} - \frac{\boldsymbol{S}_{\theta\Theta}}{\boldsymbol{R}}\frac{\partial \theta}{\partial \Theta} - \boldsymbol{S}_{\theta Z}\frac{\partial \theta}{\partial \boldsymbol{Z}}\right\}\boldsymbol{u}_r \\ & + \left\{\frac{\partial \boldsymbol{S}_{\theta R}}{\partial \boldsymbol{R}} + \frac{\boldsymbol{S}_{\theta R}}{\boldsymbol{R}} + \frac{\mathbf{1}}{\boldsymbol{R}}\frac{\partial \boldsymbol{S}_{\theta\Theta}}{\partial \Theta} + \frac{\partial \boldsymbol{S}_{\theta Z}}{\partial \boldsymbol{Z}} + \boldsymbol{S}_{rR}\frac{\partial \theta}{\partial \boldsymbol{R}} + \frac{\boldsymbol{S}_{r\Theta}}{\boldsymbol{R}}\frac{\partial \theta}{\partial \Theta} + \boldsymbol{S}_{rZ}\frac{\partial \theta}{\partial \boldsymbol{Z}}\right\}\boldsymbol{u}_\theta \\ & + \left\{\frac{\partial \boldsymbol{S}_{zR}}{\partial \boldsymbol{R}} + \frac{\boldsymbol{S}_{zR}}{\boldsymbol{R}} + \frac{\mathbf{1}}{\boldsymbol{R}}\frac{\partial \boldsymbol{S}_{z\Theta}}{\partial \Theta} + \frac{\partial \boldsymbol{S}_{zZ}}{\partial \boldsymbol{Z}}\right\}\boldsymbol{u}_z.\end{aligned} \tag{10}$$

The balance of linear momentum written in reference configuration along with the traction free boundary conditions is

$$\begin{aligned} & \boldsymbol{Div}(\mathbf{S}) = \boldsymbol{\rho}_\kappa\ddot{\boldsymbol{u}} \\ & \boldsymbol{Sn}_\kappa = \mathbf{0} \end{aligned} \tag{11}$$

where $\boldsymbol{u}$ denotes the displacement field and $\boldsymbol{n}_\kappa$ denotes the unit outward normal to the surface of the body in the reference configuration.

The problem formulation presented above is general in the sense that it holds for any kind of deformation in the cylindrical waveguides. However, we restrict our attention to longitudinal modes in this article and deal with the torsional and flexural modes in a follow up article.

3. Second harmonic generation: problem formulation for axis-symmetric longitudinal guided wave modes

First, some useful expressions for first and second Piola–Kirchhoff stress tensors will be developed. Following [12], decompose the expression for the second Piola–Kirchhoff stress into linear and nonlinear parts of second order in the displacement gradient $\mathbf{H} = \mathbf{Grad}(\mathbf{U})$ as

$$\mathbf{T_{RR}} = \mathbf{T^L_{RR}}(\mathbf{H}) + \mathbf{T^{NL}_{RR}}(\mathbf{H}) \tag{12}$$

where

$$\mathbf{T^L_{RR}}(\mathbf{H}) = \frac{\lambda}{2}\mathbf{tr}(\mathbf{H} + \mathbf{H^T})\mathbf{I} + \boldsymbol{\mu}(\mathbf{H} + \mathbf{H^T}) \tag{13}$$

and

$$\begin{aligned}\mathbf{T^{NL}_{RR}}(\mathbf{H}) = {} & \frac{\lambda}{2}\mathbf{tr}(\mathbf{H^T}\mathbf{H})\mathbf{I} + \mathbf{C}(\mathbf{tr}(\mathbf{H}))^2\mathbf{I} + \boldsymbol{\mu}\mathbf{H^T}\mathbf{H} + \mathbf{Btr}(\mathbf{H})(\mathbf{H} + \mathbf{H^T}) \\ & + \frac{\mathbf{B}}{2}\mathbf{tr}(\mathbf{H}^2 + \mathbf{H^T}\mathbf{H})\mathbf{I} + \frac{\mathbf{A}}{4}(\mathbf{H}^2 + \mathbf{H^{T^2}} + \mathbf{H}\mathbf{H^T} + \mathbf{H^T}\mathbf{H}).\end{aligned} \tag{14}$$

The first Piola–Kirchhoff stress ($\mathbf{S}$) can likewise be obtained by using Eq. (7) such that

$$\mathbf{S}(\mathbf{H}) = \mathbf{S^L}(\mathbf{H}) + \mathbf{S^{NL}}(\mathbf{H}) \tag{15}$$

where

$$\mathbf{S^L}(\mathbf{H}) = \mathbf{T^L_{RR}}(\mathbf{H}) \text{ and } \mathbf{S^{NL}}(\mathbf{H}) = \mathbf{HT^L_{RR}}(\mathbf{H}) + \mathbf{T^{NL}_{RR}}(\mathbf{H}).$$

For axis-symmetric longitudinal modes, using the notation introduced in the previous section, we have $\theta = \Theta$ and hence $\boldsymbol{u}_\mathbf{r} = \boldsymbol{u}_\mathrm{R}$, $\boldsymbol{u}_\theta = \boldsymbol{u}_\Theta$ and $\boldsymbol{u}_z = \boldsymbol{u}_\mathrm{Z}$. Also, the expression for the $\mathbf{Div}(\mathbf{S})$ is

$$\begin{aligned}\mathbf{Div}(\mathbf{S}) = {} & \left\{\frac{\partial \boldsymbol{S}_{rR}}{\partial \boldsymbol{R}} + \frac{\partial \boldsymbol{S}_{rZ}}{\partial \boldsymbol{Z}} + \frac{\boldsymbol{S}_{rR}}{\boldsymbol{R}} - \frac{\boldsymbol{S}_{\theta\Theta}}{\mathbf{R}}\right\}\boldsymbol{u}_r \\ & + \left\{\frac{\partial \boldsymbol{S}_{\theta R}}{\partial \boldsymbol{R}} + \frac{\partial \boldsymbol{S}_{\theta Z}}{\partial \boldsymbol{Z}} + \frac{\boldsymbol{S}_{\theta R}}{\mathbf{R}} + \frac{\boldsymbol{S}_{r\Theta}}{\mathbf{R}}\right\}\boldsymbol{u}_\theta \\ & + \left\{\frac{\partial \boldsymbol{S}_{zR}}{\partial \boldsymbol{R}} + \frac{\boldsymbol{S}_{zR}}{\boldsymbol{R}} + \frac{\partial \boldsymbol{S}_{zZ}}{\partial \boldsymbol{Z}}\right\}\boldsymbol{u}_z.\end{aligned} \tag{16}$$