Ultrasonics 54 (2014) 1568–1574

Contents lists available at ScienceDirect

Ultrasonics

journal homepage: www.elsevier.com/locate/ultras

A novel unsplit perfectly matched layer for the second-order acoustic wave equation

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ARTICLE INFO

Article history:
Received 14 January 2013
Received in revised form 18 March 2014
Accepted 31 March 2014
Available online 21 April 2014

Keywords:
Auxiliary-differential equation
Acoustic
Perfectly matched layer
Second-order equation

ABSTRACT

When solving acoustic field equations by using numerical approximation technique, absorbing boundary conditions (ABCs) are widely used to truncate the simulation to a finite space. The perfectly matched layer (PML) technique has exhibited excellent absorbing efficiency as an ABC for the acoustic wave equation formulated as a first-order system. However, as the PML was originally designed for the first-order equation system, it cannot be applied to the second-order equation system directly. In this article, we aim to extend the unsplit PML to the second-order equation system. We developed an efficient unsplit implementation of PML for the second-order acoustic wave equation based on an auxiliary-differential-equation (ADE) scheme. The proposed method can benefit to the use of PML in simulations based on second-order equations. Compared with the existing PMLs, it has simpler implementation and requires less extra storage. Numerical results from finite-difference time-domain models are provided to illustrate the validity of the approach.

1. Introduction

Numerical simulations of wave propagation based on partial differential equations require the truncation of the computational domain due to the limited memory and computation time of computers. The truncation can lead to artificial reflections which can ruin the simulating results. Absorbing boundary conditions (ABCs) are routinely used at the truncated boundary to attenuate the spurious reflections.

Many ABC techniques have been developed to complete this task during past years. Among them, the PML technique, first introduced by Berenger [1] in 1994, has been proven to be a very efficient scheme for the truncation of unbounded domain. Berenger's original formulation is called the split-field PML, because he artificially split the wave solutions into the sum of two new artificial field components. Though first designed for the Maxwell equations in the context of electromagnetics, the PML has now been widely used in simulations of electromagnetic, elastic, as well as acoustic wave propagations [2,3] for its excellent absorbing efficiency. Following Berenger' classical split-field scheme, a number of different implementations have been introduced [4–8], leading to the unsplit implementations of the PML. Though these implementations often lead to equivalent reflection properties, they offer different mathematical representations which may reduce implementation costs and complexity of the PML [4,7]. The auxiliary-differential-equation (ADE) [7,8] approach is one of important techniques that have facilitated implementation of the PML. It provides a more flexible representation of the PML that can be extended to higher-order methods [7].

As the classical PML was primarily designed for first-order equation system, it cannot be applied to the second-order system directly. Over past years, in the field of acoustic simulation, the PML has been mostly applied to acoustic wave equations formulated as a first-order system, rarely to equations written as a second-order system in pressure. In certain cases, a second-order system can be rewritten as a group of first-order ones and the PML can then be directly applied. However, this can lead to complex implementation, since the second-order equation system is simpler and more convenient to be used in many simulations. What is more, some well-established equations, which are often used in simulations [9–11], are not appropriate to be rewritten. Therefore, it is meaningful to extend the PML to the second-order acoustic equation. Several attempts have been reported to extend the PML for second-order equations. In 2003, Komatitsch and Tromp [12] developed a split PML for the second-order seismic wave equation. This is first effort made to apply the PML to second-order equations. This method was applied to the acoustic equation in 2010 [13]. Though Komatitsch' scheme can attenuate

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http://dx.doi.org/10.1016/j.ultras.2014.03.016
0041-624X/

propagating waves effectively, the implementation costs of his method is very expensive since one third-order and two second-order ADEs have to be solved. In 2009, Pinton et al. [11] developed an unsplit PML for the second-order acoustic equation and used it in ultrasonic imaging simulation. Though memory-efficient and seemingly simple, this mixed formulation of PML is not convenient to implement in practice for a deconvolving process involved and it is also difficult to extend to higher-order methods. In [14,15], the convolutional PML was formulated for the second-order elastic wave equation. As these two methods were designed for finite element methods (FEMs), it is not convenient to apply them in solving equations based on finite-difference time-domain (FDTD) methods or the pseudospectral time-domain (PSTD) [16] method.

In this paper, a memory-efficient unsplit PML with a very easy implementation is developed for the second-order acoustic equation. The proposed method is formulated via the complex coordinate stretching scheme and introducing ADEs. It is a method suitable for numerical methods such as the FDTD, PSTD and FEM. The remainder of the paper is organized as follows. In Section 2 the background of the proposed method is introduced, and the formulation and implementation of the proposed method is presented. The memory requirement comparison between different schemes is also carried out. In Section 3, the proposed scheme is verified and compared with classical split PML for the second-order equation in numerical experiments, followed by conclusions in Section 4.

2. Method

2.1. Background

The differential form of the linear acoustic wave equation in the frequency domain can be written as

$$\frac{1}{c^2}(j\omega)^2\hat{u} = \frac{\partial^2 \hat{u}}{\partial x^2} + \frac{\partial^2 \hat{u}}{\partial y^2} + \frac{\partial^2 \hat{u}}{\partial z^2} \tag{1}$$

where ω denotes the angular frequency, c is the speed of the sound, $\hat{u}$ represents the Fourier transform (FT) of the acoustic pressure u, and j is the imaginary unit. In the interest of notational simplicity, we only describe the solution in the x-axis using the stretched coordinate approach in [5,17]. In the PML region, Eq. (1) is formulated in complex stretched coordinates as

$$\frac{1}{c^2}(j\omega)^2\hat{u} = \frac{1}{s_x}\frac{\partial}{\partial x}\left(\frac{1}{s_x}\frac{\partial \hat{u}}{\partial x}\right) + \frac{\partial^2 \hat{u}}{\partial y^2} + \frac{\partial^2 \hat{u}}{\partial z^2} \tag{2}$$

where x-axis is stretched by

$$s_x = 1 + \sigma_x/j\omega. \tag{3}$$

Here σ_x is the damping profile across the PML region. As demonstrated in [12,13], Eqs. (2) and (3) finally lead to

$$\frac{1}{c^2}(j\omega)^2\hat{u} = -\frac{(j\omega)^2\sigma'_x}{(j\omega+\sigma_x)^3}\frac{\partial \hat{u}}{\partial x} + \frac{(j\omega)^2}{(j\omega+\sigma_x)^2}\frac{\partial^2 \hat{u}}{\partial x^2} + \frac{\partial^2 \hat{u}}{\partial y^2} + \frac{\partial^2 \hat{u}}{\partial z^2} \tag{4}$$

where $\sigma_x{}'$ is the partial derivative of σ_x with respect to x. It is difficult to derive the time-domain expression of (4) directly for its complex formulation. The general way is to introduce auxiliary variables. A split-field scheme is proposed in [12,13], where u is decomposed into 3 parts; each part is tackled separately, and the results are then combined. Though this split-field approach exhibits excellent absorbing efficiency, its numerical costs, especially the extra needed storage, are remarkably high. What is more, the splitting of pressure will often lead to cumbersome implementation when using FEM as discretization scheme [14].

2.2. The proposed method

The traditional split-field formulation of PML can actually be viewed as an ADE-based scheme, since the splitting of pressure u is also a way to introduce several new auxiliary variables to simplify the formulation. Here we present a novel ADE-based unsplit implementation for (4). In the following part, we denote $\hat{u}_i$ as the FT of u_i ($i = 1, 2, 3$). For simplicity of expression, we introduce a temporary variable p and let

$$p = j\omega + \sigma_x. \tag{5}$$

Substituting p for $j\omega$ on the right-side of (4), we have

$$\frac{1}{c^2}(j\omega)^2\hat{u} = -\frac{(p-\sigma_x)^2\sigma'_x}{p^3}\frac{\partial \hat{u}}{\partial x} + \frac{(p-\sigma_x)^2}{p^2}\frac{\partial^2 \hat{u}}{\partial x^2} + \frac{\partial^2 \hat{u}}{\partial y^2} + \frac{\partial^2 \hat{u}}{\partial z^2}. \tag{6}$$

Enforcing a partial expansion scheme on the right-side of the above equation and arrange its terms according to the order of p, Eq. (6) becomes

$$\begin{aligned}\frac{1}{c^2}(j\omega)^2\hat{u} = {}& \frac{\partial^2 \hat{u}}{\partial x^2} + \frac{\partial^2 \hat{u}}{\partial y^2} + \frac{\partial^2 \hat{u}}{\partial z^2} - \left(\frac{\sigma'_x}{p}\frac{\partial \hat{u}}{\partial x} + \frac{2\sigma_x}{p}\frac{\partial^2 \hat{u}}{\partial x^2}\right) \\ & + \left(\frac{\sigma_x^2}{p^2}\frac{\partial^2 \hat{u}}{\partial x^2} + \frac{2\sigma_x\sigma'_x}{p^2}\frac{\partial \hat{u}}{\partial x}\right) - \frac{\sigma'_x\sigma_x^2}{p^3}\frac{\partial \hat{u}}{\partial x}\end{aligned} \tag{7}$$

Introducing an auxiliary u_1 and rearranging the above equation, we rewrite (7) as

$$\frac{1}{c^2}(j\omega)^2\hat{u} = \frac{\partial^2 \hat{u}}{\partial x^2} + \frac{\partial^2 \hat{u}}{\partial y^2} + \frac{\partial^2 \hat{u}}{\partial z^2} - \hat{u}_1 \tag{8}$$

where u_1 satisfies

$$\hat{u}_1 = \frac{\sigma'_x}{p}\frac{\partial \hat{u}}{\partial x} + 2\frac{\sigma_x}{p}\frac{\partial^2 \hat{u}}{\partial x^2} - \sigma_x\left(\frac{2\sigma'_x}{p^2}\frac{\partial \hat{u}}{\partial x} + \frac{\sigma_x}{p^2}\frac{\partial^2 \hat{u}}{\partial x^2}\right) + \frac{\sigma_x^2\sigma'_x}{p^3}\frac{\partial \hat{u}}{\partial x}. \tag{9}$$

Multiplying both sides of (9) with p, we have

$$p\hat{u}_1 = \sigma'_x\frac{\partial \hat{u}}{\partial x} + 2\sigma_x\frac{\partial^2 \hat{u}}{\partial x^2} - \sigma_x\hat{u}_2 \tag{10}$$

where

$$\hat{u}_2 = \left(\frac{2\sigma'_x}{p}\frac{\partial \hat{u}}{\partial x} + \frac{\sigma_x}{p}\frac{\partial^2 \hat{u}}{\partial x^2}\right) - \frac{\sigma_x\sigma'_x}{p^2}\frac{\partial \hat{u}}{\partial x}. \tag{11}$$

Multiplying both sides of the above equation with p, Eq. (11) becomes

$$p\hat{u}_2 = 2\sigma'_x\frac{\partial \hat{u}}{\partial x} + \sigma_x\frac{\partial^2 \hat{u}}{\partial x^2} - \sigma_x\hat{u}_3 \tag{12}$$

where

$$\hat{u}_3 = \frac{\sigma'_x}{p}\frac{\partial \hat{u}}{\partial x}. \tag{13}$$

According to (13), we can easily get

$$p\hat{u}_3 = \sigma'_x\frac{\partial \hat{u}}{\partial x}. \tag{14}$$

Inserting (5) into (8), (10), (12), and (14), we obtain the following frequency domain equations in Cartesian coordinates

$$(j\omega+\sigma_x)\hat{u}_3 = \sigma'_x\frac{\partial \hat{u}}{\partial x} \tag{15a}$$

$$(j\omega+\sigma_x)\hat{u}_2 = 2\sigma'_x\frac{\partial \hat{u}}{\partial x} + \sigma_x\frac{\partial^2 \hat{u}}{\partial x^2} - \sigma_x\hat{u}_3 \tag{15b}$$

$$(j\omega+\sigma_x)\hat{u}_1 = \sigma'_x\frac{\partial \hat{u}}{\partial x} + 2\sigma_x\frac{\partial^2 \hat{u}}{\partial x^2} - \sigma_x\hat{u}_2 \tag{15c}$$