Ultrasonics 51 (2011) 123–130

Contents lists available at ScienceDirect

Ultrasonics

journal homepage: www.elsevier.com/locate/ultras

Propagation of elastic waves in an anisotropic functionally graded hollow cylinder in vacuum

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ARTICLE INFO

Article history:
Received 19 February 2010
Received in revised form 1 July 2010
Accepted 2 July 2010
Available online 7 August 2010

Keywords:
Elastic wave propagation
Radially graded tube
Stroh's formalism
Waveguide
In vitro bone characterization

ABSTRACT

As a non-destructive, non-invasive and non-ionizing evaluation technique for heterogeneous media, the ultrasonic method is of major interest in industrial applications but especially in biomedical fields. Among the unidirectionally heterogeneous media, the continuously varying media are a particular but widespread case in natural materials. The first studies on laterally varying media were carried out by geophysicists on the Ocean, the atmosphere or the Earth, but the teeth, the bone, the shells and the insects wings are also functionally graded media. Some of them can be modeled as planar structures but a lot of them are curved media and need to be modeled as cylinders instead of plates. The present paper investigates the influence of the tubular geometry of a waveguide on the propagation of elastic waves. In this paper, the studied structure is an anisotropic hollow cylinder with elastic properties (stiffness coefficients c_{ij} and mass density ρ) functionally varying in the radial direction. An original method is proposed to find the eigenmodes of this waveguide without using a multilayered model for the cylinder. This method is based on the sextic Stroh's formalism and an analytical solution, the matricant, explicitly expressed under the Peano series expansion form. This approach has already been validated for the study of an anisotropic laterally-graded plate (Baron et al., 2007; Baron and Naili, 2010) [6,5]. The dispersion curves obtained for the radially-graded cylinder are compared to the dispersion curves of a corresponding laterally-graded plate to evaluate the influence of the curvature.

Preliminary results are presented for a tube of bone in vacuum modelling the *in vitro* conditions of bone strength evaluation.

1. Introduction

The observation of natural media and particularly of living tissues is a great source of inspiration for scientists. As an example, they develop industrial Functionally Graded Materials (FGM) in the 80s which reproduce a characteristic observed in natural media such as wood, bone or shells. The continuous variation of the mechanical properties of these materials reveals interesting mechanical behavior particularly exploited in high-technology and biomedical applications [48,19,26]. As a consequence, the non-destructive characterization of FGM structures became a key issue: first, to better understand the natural mechanisms observed and second, to guide the conception of groundbreaking FGM. Surface and guided waves are significant information source in non-destructive testing and evaluation of complex structures. A lot of works detailed the behavior of the guided waves in isotropic or anisotropic plates [39,23,46,3]. Also, elastic wave propagation in cylindrical structures formed from material with lower anisotropy than orthotropy has been the subject of numerous theoretical and experimental investigations widely published [30,50,20]. For anisotropic material, the complexity of the problem relies on the fact that the boundary problem do not permit solution in cylindrical functions except in some particular configurations [29].

In this work, we solve the wave equation in an anisotropic waveguide with one-direction heterogeneity using a general method based on the sextic Stroh's formalism [44]. It takes into account the unidirectional continuous variation of the properties of the waveguide without using a multilayered model. It is based on the knowledge of an analytical solution of the wave equation, the matricant, explicitly expressed *via* the Peano series expansion [6]. The accuracy of the numerical evaluation of this solution and its validity domain are perfectly managed [4,49]. One of the advantages of knowing an analytical solution with respect to purely numerical methods is to control all the physical parameters and to interpret more easily the experimental data which result from the interaction and coupling of numerous physical phenomena.

In this paper, a sample of long bone is considered as an example of anisotropic functionally graded tube. The material is multiscale

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doi:10.1016/j.ultras.2010.07.001

and the discrete variations of its microscopic properties (bone matrix elasticity, micro-architecture, etc.) [9,8,41] is assumed to induce continuous profiles of macroscopic properties in the radial direction [7,16,5]. Obviously, the mechanical behavior of bone depends on several parameters (microstructure, elasticity and geometry). The curvature is part of them and its influence remains unclear.

We first present the method and its setup for the cylindrical waveguide; this method has been validated by comparing our results to the dispersion curves obtained from classical schemes on homogeneous and functionally graded waveguides. Some advantages of the method are underlined: (i) general anisotropy may be taken into account for cylindrical structures; (ii) the influence of the property gradient on the mechanical behavior of the waveguide may be investigated; (iii) the influence of the curvature on the propagation of elastic waves may be evaluated.

2. Methods

We consider an elastic tube of thickness t placed in vacuum.

The radius r varies from a_0 to a_q, respectively, the inner and outer radius of the tube (Fig. 1). The elastic cylinder is supposed to be anisotropic and is liable to present continuously varying properties along its radius ($\mathbf{e}_r$-axis). These mechanical properties are represented by the stiffness tensor $\mathbb{C} = \mathbb{C}(r)$ and the mass density $\rho = \rho(r)$.

2.1. System equations

The momentum conservation equation associated with the constitutive law of linear elasticity (Hooke's law) gives the following equations:

$$\begin{cases} \operatorname{div} \boldsymbol{\sigma} = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2}, \\ \boldsymbol{\sigma} = \frac{1}{2} \mathbb{C} (\operatorname{grad} \mathbf{u} + \operatorname{grad}^T \mathbf{u}), \end{cases} \tag{1}$$

where $\mathbf{u}$ is the displacement vector and $\boldsymbol{\sigma}$ the stress tensor.

We are seeking the solutions of wave equation for displacement ($\mathbf{u}$) and radial traction-vector ($\boldsymbol{\sigma}_r = \boldsymbol{\sigma}.\mathbf{e}_r$) expressed in the cylindrical coordinates (r,θ,z) with the basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$:

$$\begin{aligned} \mathbf{u}(r,\theta,z;t) &= \mathbf{U}^{(n)}(r) \exp \imath (n\theta + k_z z - \omega t), \\ \boldsymbol{\sigma}_r(r,\theta,z;t) &= \mathbf{T}^{(n)}(r) \exp \imath (n\theta + k_z z - \omega t); \end{aligned} \tag{2}$$

with k_z the axial wavenumber and n the circumferential wavenumber.

We distinguish two types of waves propagating in a cylindrical waveguide: the *circumferential waves* and the *axial waves*. The *circumferential waves* are the waves traveling in planes perpendicular to the axis direction. They correspond to $u_z(r) = 0$ ($\forall r$), $k_z = 0$ and $n = k_\theta a_q$. The *axial waves* are the waves traveling along the axis direction, the circumferential wavenumber is an integer $n = 0,1,2,\ldots$. Among the *axial waves*, we distinguish three types of modes numbered with two parameters (n,m) representing the circumferential wavenumber and the order of the branches: longitudinal (L), flexural (F) and torsional (T) modes. The longitudinal modes are axially symmetric ($n = 0$), they are noted $L(0,m)$ and sometimes called the breathing modes [3]. For the two other types of modes, two classifications have been proposed in the literature. Meitzler [28] and Zemanek [50], following Gazis work [14], restricted the T-modes to axially symmetric circumferential fundamental modes $T(0,m)$ and the modes with non-zero n are considered as F-modes $F(n,m)$. An other way is proposed by Nishino and colleagues [34] who considered that the n-parameter of the T-modes is not limited to zero. Consequently they can associate "their" L- and F-modes to the Lamb waves (coupled bulk longitudinal and bulk shear-vertical waves) in the plate and the T-modes to the bulk horizontal-shear waves propagating in the plate. This classification may be relevant in the case of isotropic media but it becomes invalid for anisotropic media for which the flexural waves are longitudinal shear waves [14].

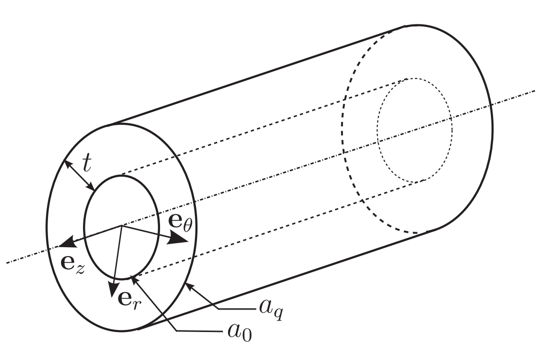

Fig. 1. Geometrical configuration of the waveguide.

In this paper, only the *axial waves* are investigated and the usual classification of Gazis [14] is used.

2.2. A closed-form solution: the matricant

Introducing the expression (2) in Eq. (1), we obtain the wave equation under the form of a second-order differential equation with non-constant coefficients. In the general case, there is no analytical solution to the problem thus formulated. The most current methods to solve the wave equation in unidirectionally heterogeneous media are derived from the Thomson–Haskell method [45,18]. These methods are appropriate for multilayered structures [22,27,47,21]. But, for continuously varying media, these techniques mean to replace the continuous profiles of properties by step-wise functions. Thereby the studied problem becomes an approximate one, even before the resolution step; the accuracy of the solution as its validity domain are hard to evaluate. Moreover, the multilayered model of the functionally graded waveguide creates some "virtual" interfaces likely to induce artefacts. Lastly, for generally anisotropic cylinders, the solutions cannot been expressed analytically even for homogeneous layers [29,32,43].

In order to deal with the exact problem, that is to keep the continuity of the properties variation, the wave equation is written under the sextic Stroh's formalism [44] in the form of an ordinary differential equations system with non-constant coefficients for which an analytical solution exists: the matricant [4]. Another method relies on the Legendre's polynomial as explained and used in [25,11,13,12].

2.2.1. Hamiltonian form of the wave equation

In the Fourier domain, the wave equation may be written as:

$$\frac{d}{dr}\boldsymbol{\eta}(r) = \frac{1}{r}\mathbf{Q}(r)\boldsymbol{\eta}(r). \tag{3}$$

The components of the state-vector $\boldsymbol{\eta}(r)$ are the three components of the displacement and the three components of the stress traction in the cylindrical coordinates, and the matrix $\mathbf{Q}(r)$ contains all the information about the heterogeneity: it is expressed from stiffness coefficients of the waveguide in the cylindrical coordinates and from the two acoustical parameters: axial or circumferential wavenumbers (k_z, or n) and the angular frequency ω. The detailed expression of $\mathbf{Q}(r)$ is given in Appendix A for the case of a material with orthorhombic crystallographic symmetry but it can be expressed for any type of anisotropy [42].