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● *Original Contribution*

WAVE SIMULATION IN BIOLOGIC MEDIA BASED ON THE KELVIN-VOIGT FRACTIONAL-DERIVATIVE STRESS-STRAIN RELATION

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Abstract—The acoustic behavior of biologic media can be described more realistically using a stress-strain relation based on fractional time derivatives of the strain, since the fractional exponent is an additional fitting parameter. We consider a generalization of the Kelvin-Voigt rheology to the case of rational orders of differentiation, the so-called Kelvin-Voigt fractional-derivative (KVFD) constitutive equation, and introduce a novel modeling method to solve the wave equation by means of the Grünwald-Letnikov approximation and the staggered Fourier pseudospectral method to compute the spatial derivatives. The algorithm can handle complex geometries and general material-property variability. We verify the results by comparison with the analytical solution obtained for wave propagation in homogeneous media. Moreover, we illustrate the use of the algorithm by simulation of wave propagation in normal and cancerous breast tissue. (E-mail: jcarcione@inogs.it) © 2011 World Federation for Ultrasound in Medicine & Biology.

Key Words: **Biologic media, Anelasticity, Fractional derivatives, Waves, Kelvin-Voigt, Dissipation.**

INTRODUCTION

The description of the physical and chemical behavior of living matter by using fractional derivatives has recently gained increasing interest in the medical community for the characterization of pathologies. New imaging methods are based on fractional stress-strain relations to interpret data obtained with ultrasound elastography (Coussot et al. 2009), where the shear and Young's moduli are the relevant elastic constants. Fractional derivatives have been used to describe the viscoelastic characterization of liver (Taylor et al. 2002), the flow of small molecules across biologic membranes (Caputo and Cametti 2008a 2008b; Caputo et al. 2009; Cesarone et al. 2005) and breast-tissue attenuation in ultrasound propagation (Bounaïm et al 2007; Bounaïm and Chen 2008). Magin et al. (2009) solved the Bloch equation, which relates a macroscopic model of magnetization to applied radio-frequency in gradient and static magnetic fields, to detect and characterize neurodegenerative, malignant and ischemic diseases. The overview of the methods based on fractional calculus and used in bioengineering is given in Magin (2006).

Stress-strain relations based on fractional derivatives provide a suitable model of wave attenuation in anelastic media. Bland (1960), Caputo (1967), Kjartansson (1979) and Caputo and Mainardi (1971) described the anelastic behavior of general materials over wide frequency ranges by using fractional derivatives, in particular considering propagation with constant-Q characteristics. In this case, Mainardi and Tomirotti (1997) obtained the one-dimensional (1-D) Green's function based on the Mittag-Leffler functions.

One of the most used stress-strain relations for biologic tissues is the Kelvin-Voigt fractional-derivative (KVFD) model, first introduced by Caputo (1981) to model underground nuclear explosions (Taylor et al. 2002; Kiss et al. 2004; Coussot et al. 2009; Kelly and McGough 2009). Since then, several authors studied and used the properties of the KVFD model. Schiessel et al. (1995) obtained analytical solutions in terms of FoxH- and Mittag-Leffer functions. Eldred et al. (1995) fit experimental data for both rubbery and a glassy viscoelastic material. Important applications include biomedical engineering. Zhang et al. (2008) showed that stress relaxation tests on prostate samples produced repeatable results that fit a viscoelastic KVFD model.

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Similarly, Coussot et al. (2009) showed that the KVFD relation can characterize the viscoelastic properties of hydropolymers, in particular normal and cancerous breast tissues, stating that this approach may ultimately be applied to tumor differentiation. Here, we consider the KVFD stress-strain relation that is based on three free parameters to describe the viscoelastic behavior of biologic media. Combining this relation with Newton's equation yields the so-called "Caputo wave equation" (Caputo 1967) studied by Holm and Sinkus (2010).

Numerical simulations of wave propagation in an axisymmetric three-dimensional (3-D) domain, based on the Caputo wave equation, have been performed by Wismer (2006) in the low-frequency range, using a finite element method. Regarding other numerical simulations in more than one dimension, Caputo and Carcione (2010) generalized the one-term stress-strain relation (spring or dashpot) to the fractional case, which includes Hooke's law at the lower limit of the fractional order of differentiation and the constitutive relation of a dashpot at the corresponding upper limit. In this case, these authors considered a spectrum of orders of differentiation. The numerical simulation of two-dimensional (2-D) seismic compressional (P)-wave propagation in heterogeneous media for one order of differentiation has been implemented by Carcione et al. (2002), while the 3-D P-S (shear) case has been developed and solved numerically in two dimensions by Carcione (2009). To our knowledge, there are a few works that use the KVFD approach to solve the wave equation in more than one dimension. Besides Wismer (2006), Dikmen (2005) applied the model to simulate 2-D seismic wave attenuation in soil structures and employed the finite-element algorithm to solve the wave equation. In bioacoustics, there is the work of Bounaïm et al (2007) and Bounaïm and Chen (2008), who used a finite-element method to perform 2-D numerical simulations to investigate the detectability of breast tumors. They have used a fractional Laplacian (Chen and Holm 2004) instead of fractional time derivates. It is important to point out that attenuation can also be described by using spatial fractional derivatives (Carcione 2010; Treeby and Cox 2010).

Here, we propose to solve the differential equations with a direct method, where the spatial derivatives are computed by using the staggered Fourier pseudospectral method (*e.g.*, Carcione 2007; Caputo and Carcione 2010). Fractional time derivatives are computed with the Grünwald-Letnikov (GL) approximation (Grünwald 1867; Letnikov 1868; Caputo 1967; Carcione et al. 2002), which is an extension of the standard finite-difference approximation for derivatives of integer order.

In the first part of this work, we introduce the stress-strain relation and calculate the complex moduli, phase velocities and attenuation and quality factors vs. frequency. We then recast the wave equation in the time-domain in terms of fractional derivatives and obtain the GL approximation. The model is discretized on a mesh and the spatial derivatives are calculated with the Fourier method by using the fast Fourier transform. Finally, we perform numerical experiments in breast fatty tissue and breast cancer to study the influence of anelasticity on the wave field. The experiments simulate the clinical amplitude/velocity reconstruction imaging (CARI) technique, which is an ultrasonic method for the detection of breast cancer (Richter 1994). It is based on the reflection of waves at a metallic plate. In CARI, reflection through the breast without the tumor shows a uniform pattern, while in the presence of tumor the field arrives earlier and shows more attenuation.

MATERIALS AND METHODS

The stress-strain relation

Attenuation can be described by means of additional first-order time differential equations (*e.g.*, Carcione et al. 1988; Wojcik et al. 1999) or by using power laws in the form of fractional derivatives. This approach approximates better the behavior of real media. Caputo and Mainardi (1971) describe the anelastic behavior of many materials over wide frequency ranges by using fractional derivatives. We consider the generalization of the Kelvin-Voigt stress (σ)-strain (ϵ) relation as

$$\sigma = M\epsilon + \eta \frac{\partial^q \epsilon}{\partial t^q},\ 0 \leq q \leq 1, \tag{1}$$

where M is the stiffness, and η is a pseudo-viscosity, which is a stiffness for q = 0 and a viscosity for q = 1. The limits $q = 0$ and $q = 1$ give Hooke's law and the constitutive relation of a spring in parallel connection with a dashpot, *i.e.*, the Kelvin-Voigt model (Carcione 2007).

In the frequency domain, we obtain

$$\sigma = \overline{M}\varepsilon, \tag{2}$$

where

$$\overline{M} = M + \eta(\mathrm{i}\omega)^q \tag{3}$$

is the complex stiffness, with ω the angular frequency. We may write (Carcione 2009)

$$\eta = \eta_0 \omega_0^{-q}, \tag{4}$$

where ω_0 is a reference frequency. Then,

$$\overline{M} = M + \eta_0 \left(\frac{\mathrm{i}\omega}{\omega_0}\right)^q. \tag{5}$$

Note that η has the units [Pa s^q]. The complex modulus $\overline{M}$ given by eqn (5) reduces to the real modulus