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Spectra that behave like power-laws are not necessarily power-laws

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Abstract

It is shown that measured power spectral densities (spectra) that closely resemble power-law spectra may, in fact, have mathematical forms that are not power laws in the mathematical sense. If power spectral estimates show a good fit to a straight line on a log–log plot over a finite frequency range, that is not sufficient evidence to conclude that the mathematical form of the spectrum is, in fact, a power-law over that range. It is also pointed out that to accurately fit a power-law function to experimental data using linear least squares techniques in log–log space, as is often done in practice, it is essential that the data is uniformly distributed along the abscissa in log-space (in the stochastic sense) or, otherwise, the data must be linearly interpolated onto a uniform grid to ensure that the data employed in the fitting procedure is equally weighted along the abscissa. These two important points are not widely appreciated by researchers in the field and the pitfalls associated with commonly used fitting techniques are often overlooked in the analysis of solar wind data.

Keywords: Solar wind; Power-law spectra; Data analysis techniques

1. Introduction

A significant advance in solar wind science is the recent discovery that the power spectral density (the spectrum) of the total energy of MHD scale fluctuations in the "inertial range" has a spectral index near 3/2, on average (Podesta, 2013a,b, and references therein). This is a consequence of the fact that the spectrum of magnetic field fluctuations is usually steeper than the spectrum of plasma velocity fluctuations, indicating an excess of magnetic energy over kinetic energy scale by scale—one of the characteristic features of MHD turbulence (Chen et al., 2013). The spectral index of the trace spectrum of the magnetic field is near 5/3, on average. The spectrum of the *total energy* is the sum of the trace spectra of the magnetic field fluctuations and the plasma velocity fluctuations, both expressed in the same physical units, and the value 3/2 observed in the solar wind is consistent with that observed in simulations of homogeneous incompressible MHD turbulence (Boldyrev et al., Nov. 2011). In the solar wind, these spectra behave like power laws over more than two or three decades in frequency and/or wavenumber. Similar power-law spectra are observed in different branches of science.

Estimates of power spectral densities obtained from experimental data often behave like power-laws over finite frequency ranges. But this does not necessarily imply or demonstrate that a spectrum is, in fact, a power-law spectrum over the indicated frequency range, that is, a spectrum of the form $S(\omega) = A/\omega^{\alpha}$, where A and α are positive constants. For example, a spectrum of the form $S(\omega) = \omega^{-\alpha}\log(\omega)$ resembles a power-law spectrum on a log–log plot and may easily be mistaken for a power law spectrum by a scientist analyzing experimental data, even though it is not a pure power-law spectrum in the mathematical sense. Similarly, a gradual curvature of the spectrum can often go unnoticed, even by experienced researchers. Therefore, it is important to keep in mind that what "looks like" or "behaves like" a power-law spectrum

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on a log–log plot, may not be a true power-law spectrum in the mathematical sense of the term. The first goal of this study is to demonstrate this important point.

The second goal is to demonstrate why it is necessary to use uniformly distributed abscissa when estimating the power law exponent of a function $y = f(x)$ from a plot of $\log(x)$ versus $\log(y)$ using linear least squares techniques, a technique commonly used in space physics and other fields.

2. Asymptotic behavior of spectra

The rate of growth of the function $\log(x)$ as $x \to \infty$ is less than the rate of growth of the function x^ϵ for any $\epsilon > 0$. Consequently, the following three functions are, in order of increasing growth,

$$1, \quad \log(x), \quad x^\epsilon, \quad \epsilon > 0. \tag{1}$$

With the substitution $x = \log(t)$, it follows that for any constant $\beta > 0$, the rate of growth of the function $\log[\log(x)]$ is less than the rate of growth of the function $[\log(x)]^\beta$ which is, in turn, less than the rate of growth of the function x^ϵ for any $\epsilon > 0$. Therefore, for any positive constant $\alpha > 0$, the following four functions are listed in order of increasing growth:

$$x^{-\alpha}, \quad x^{-\alpha}\log[\log(x)], \quad x^{-\alpha}[\log(x)]^\beta, \quad x^{\epsilon-\alpha}, \tag{2}$$

and this ordering holds for any $\epsilon > 0$. This shows that given two decreasing power-law functions with arbitrarily close power law indices α and $\alpha - \epsilon$, there exist functions with rates of growth greater than $x^{-\alpha}$ and less than $x^{-(\alpha-\epsilon)}$ that do not behave asymptotically like a power law.

From a practical point of view, the functional dependence of functions like those just discussed are often indistinguishable from power laws if the numerical values of the function are given on a finite interval but the mathematical definition or analytical expression are not given or not known. For example, consider the functions

$$y_1(\omega) = 200\omega^{-3/2} \quad \text{and}$$
$$y_2(\omega) = 100\omega^{-5/3}\log(1+\omega), \tag{3}$$

where $\omega > 0$. These two functions are almost indistinguishable to the eye when viewed on a log–log plot over a range spanning three decades, $10 < \omega < 10^4$, as shown in the left-hand plot in Fig. 1. This is remarkable considering that one function has a power-law index of 3/2 while the other has a leading coefficient with a power-law index of 5/3. These two functions differ by roughly 10% over the entire frequency range, as shown in the right-hand plot in Fig. 1.

If one performs a linear least squares fit to the data in this particular frequency interval using 150 points that are equally spaced along the abscissa $\log(\omega)$, as described in greater detail in Section 4, one finds that the fit to the function $y_1(\omega)$ has a spectral index $\alpha = 1.50$ whereas the fit to the function $y_2(\omega)$ has a spectral index $\alpha = 1.48$. It would be incorrect to conclude from this fitting procedure that the data for y_2 comes from a power-law spectrum with a power-law index of approximately $\alpha = 1.48$, although it certainly does behave that way over this particular frequency interval. The frequency spectrum $y_2(\omega)$ is not a power-law spectrum.

3. Power spectral densities in nature

Power-law spectra are believed to describe a wide range of naturally occurring phenomena, although the applicable frequencies are usually limited to a finite range. Consequently, power-law spectra frequently arise in physical models of natural phenomena. The Ornstein–Uhlenbeck process that models the velocity of particles undergoing Brownian motion in Einstein's theory has an autocovariance function

$$C(\tau) = \sigma^2 e^{-\lambda\tau}, \quad \tau > 0, \tag{4}$$

where σ and λ are positive constants and $C(-\tau) = C(\tau)$. The power spectral density, the Fourier transform of $C(\tau)$, is

$$S(\omega) = \sigma^2 \frac{2\lambda}{\omega^2 + \lambda^2}. \tag{5}$$

This spectrum has an asymptotic spectral index $\alpha = 2$.

Mathematically, any non-negative real valued function $S(\omega)$ with the properties $S(-\omega) = S(\omega)$ and $\int_{-\infty}^{\infty} S(\omega)\, d\omega < \infty$ is the spectrum of a stationary stochastic process (Davenport and Root, 1958, page 106) and, therefore, the unusual kinds of asymptotic behavior studied in the last section are physically realizable. Nevertheless, the extent to which they represent physical processes in nature is not widely known. Examples of such spectra are easily constructed, as will now be shown.

Consider the autocovariance function

$$C(\tau) = \sigma^2\left[e^{-\lambda\tau} - (\lambda\tau)E_1(\lambda\tau)\right], \quad \tau > 0, \tag{6}$$

which is defined for negative τ by $C(-\tau) = C(\tau)$. Here, $\lambda > 0$ and

$$E_1(x) = \int_x^\infty \frac{e^{-t}}{t}\, dt, \quad x > 0, \tag{7}$$

is the exponential integral (Abramowitz and Stegun, 1972, Chapter 5). The corresponding spectrum is the Fourier transform of $C(\tau)$ given by

$$S(\omega) = \frac{\lambda\sigma^2}{\omega^2}\log\left(1 + \frac{\omega^2}{\lambda^2}\right). \tag{8}$$

As $\omega \to \infty$, the spectrum (8) behaves like $\log(\omega)/\omega^2$, which is not a power-law. However, on a log–log plot $S(\omega)$ looks like a power-law spectrum because the factor $\log(\omega)$ changes much more slowly than the factor ω^{-2} (see Fig. 2).

A natural generalization of the spectrum (8) is

$$S(\omega) = \frac{\sigma^2}{\lambda}\left(\frac{\lambda^2}{\omega^2}\right)^\nu \log\left[1 + \left(\frac{\omega^2}{\lambda^2}\right)^\nu\right], \tag{9}$$