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Two algorithms to compute Hansen-like coefficients with respect to the eccentric anomaly

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Abstract

Considering a point of polar coordinates (r, v) on an elliptic orbit of semi-major axis a, we set up and compare two algorithms based on recurrence relations to compute the Hansen-like coefficients $Z_s^{n,m}$, which are the coefficients of the expansion of $(r/a)^n \exp imv$ in Fourier series of the eccentric anomaly. Both Hansen-like coefficients and their derivatives with respect to the eccentricity are considered, with a special focus on the case $0 \leqslant |m| \leqslant n$ arising in the expression of the gravity potential due to a body external to the elliptic orbit. We provide two efficient algorithms to compute a table of coefficients with a simple recursive process. One algorithm uses some recurrence relations linking directly to the $Z_s^{n,m}$ whereas the other algorithm involves the generalized Laplace coefficients $b_{p,r}^k$ (Laskar, 2005). Numerical behavior of the algorithms is investigated for low and high eccentricities. Both algorithms provide a relative accuracy better than 10^{-14} for $n \leqslant 30$. Also, they are at least 10 time faster than an algorithm based on the FFT method (Klioner et al., 1997).

Keywords: Hansen-like coefficients; Recurrence relations; Eccentric anomaly; Highly elliptical orbit; Third-body; Closed-form

1. Introduction

The development of some functions of the Cartesian coordinates in trigonometric series of angular orbital elements is a corner stone of celestial mechanics (e.g. Tisserand, 1889). This is particularly useful in order to express the disturbing function in terms of orbital elements so as to construct analytical theories by means of averaging transformations. The paradigm of this problem is the use of the Hansen coefficients $X_s^{n,m}$ such that

$$\left(\frac{r}{a}\right)^n \exp imv = \sum_{s=-\infty}^{\infty} X_s^{n,m}(e) \exp is\, M, \tag{1.1}$$

where n and $m \in \mathbb{Z}$, $i = \sqrt{-1}$, r is the radius vector, a the semi-major axis, e the eccentricity, M the mean anomaly and v the true anomaly.

The main advantage of this development is that it is expressed as a function of the mean anomaly. The mean anomaly is time linear in case of unperturbed two-body problem and belongs to the set of Delaunay canonical variables. Unfortunately, the convergence of this Fourier series can be very slow as soon as the eccentricity of the orbit is no longer small. It even becomes useless when the eccentricity is close to 1. In such a situation, a valuable solution is to use finite Fourier series of the true anomaly if $n \leqslant 0$ (as done by Brouwer (1959) for the perturbation by the gravity potential of the central body), or finite Fourier series of the eccentric anomaly E if $n > 0$ (e.g. for the development of the perturbation by an external body). In the latter case, the so-called Hansen-like coefficients $Z_s^{n,m}$ (Brumberg, 1995) are involved:

$$\left(\frac{r}{a}\right)^n \exp imv = \sum_{s=-\infty}^{\infty} Z_s^{n,m} \exp is\, E. \tag{1.2}$$

We have used these coefficients to develop the expression of the disturbing gravity potential due to an external body

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as a function of the eccentric anomaly (Lion and Métris, 2011) and we are constructing an analytical theory of the motion using this expression. This formulation has the advantage to provide an exact expression without truncation, excepted with respect to the ratio a/a' between the satellite and the third body semi-major axes. Of course, as in the Brouwer's case, the theory in closed form is more difficult to elaborate than a theory using truncated expressions in mean anomaly. We will present our solution in a forthcoming paper.

When a theory using the $Z_s^{n,m}$ coefficients has been developed, the question of their evaluation arises. Two main options are possible: evaluating directly the coefficients from their analytical expression or using recurrence relations. Generally, direct expressions are faster for computing a small number of coefficients whereas recurrence relations are more efficient to evaluate a full table of coefficients. An alternative method consists in computing the Fourier coefficients by means of Fast Fourier Transform (FFT) (Klioner et al., 1997). To express the Hansen-like coefficients $Z_s^{n,m}$, we have at our disposal their formulation by means of hypergeometric functions (Brumberg, 1995) or by means of generalized Laplace coefficients (Laskar, 2005). To our knowledge, there are few works about recurrence relations for the $Z_s^{n,m}$ coefficients. An attempt was published by Vinh (1970) who established basic recurrence formulae and presented a simple algorithm to compute the series expansions (1.2). However, his numerical scheme is not stable for large value of m due to the divisor e.

The aim of this article is to establish precise and explicit algorithms to compute the coefficients $Z_s^{n,m}$ and to test their efficiency. More precisely, we propose and compare two algorithms: the first (named Z-algorithm in the following) uses some recurrence relations linking directly these coefficients whereas the second algorithm (named b-algorithm) involves the generalized Laplace coefficients $b_{p,r}^k$ (Laskar, 2005). The paper is organized as follows. In Section 2, we recall the expressions of the Fourier coefficients $Z_s^{n,m}$ in terms of hypergeometric functions. We present in Section 3 the recurrence relations used for the Z-algorithm in Section 4 and we discuss the methods for computing derivatives. In Section 4, we focus on the computation of a table of coefficients $Z_s^{n,m}$ and their derivatives in the case $0 \leqslant |m| \leqslant n$. Finally, the efficiency and the stability of the algorithms are investigated in Section 5 for low and high eccentricities ($e = 0.01$ and $e = 0.8$) and for large indices ($0 \leqslant n \leqslant 30$ and $0 \leqslant m \leqslant n$).

2. Expression of Hansen-like coefficients

Let

$$\Phi_{n,m} = \left(\frac{r}{a}\right)^n \exp \mathrm{i}mv \tag{2.1}$$

be the elliptic motion functions and

$$\eta = \sqrt{1-e^2}, \quad \beta = \frac{e}{1+\eta}, \tag{2.2}$$

the classical parameters (see Tisserand, 1889) linked by the relations

$$\beta^2 = \frac{1-\eta}{1+\eta}, \quad e = \frac{2\beta}{1+\beta^2}, \quad \eta = \frac{1-\beta^2}{1+\beta^2}. \tag{2.3}$$

The Fourier coefficients in (1.2) depend on e or β and admit an integral representation,

$$Z_s^{n,m} = \frac{1}{2\pi}\int_0^{2\pi} \Phi_{n,m} \exp(-\mathrm{i}sE)\mathrm{d}E. \tag{2.4}$$

Due to the fact that v is an odd function of E, if we change m by $-m$ and s by $-s$ in (1.2), we get directly the symmetry:

$$Z_{-s}^{n,-m} = Z_s^{n,m}, \tag{2.5}$$

as in the case of the classical Hansen coefficients $X_s^{n,m}$.

Let us introduce the Gaussian hypergeometric series $F(a,b;c;z)$ (Abramowitz and Stegun, 1972):

$$F(a,b;c;z) = F(b,a;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k(1)_k} z^k, \tag{2.6}$$

where the Pochhammer symbol $(a)_k$ is defined for non-negative integers k by

$$(a)_0 = 1, \quad (1)_k = k!, \tag{2.7a}$$
$$(a)_k = a(a+1)\dots(a+k-1) = (a+k-1)(a)_{k-1}. \tag{2.7b}$$

An important characteristic of the hypergeometric functions is that if a or b is a negative or null integer and $c \in \mathbb{N}^*$, the series (2.6) reduces to a finite polynomial. Apart from this latter case, the radius of convergence of the hypergeometric series is 1, as this is easily seen using the d'Alembert's criterion.

A general and convenient expression for the Z-functions using the hypergeometric function is given by Brumberg (1995), (see Eqs. (2.3.40) and (2.3.42)). The original expression in terms of β can be written in the following form

$$Z_s^{n,m}(\beta) = A_s^{n,m}(\beta)V_s^{n,m}(\beta), \tag{2.8}$$

where

$$A_s^{n,m} = \frac{(-n-m)_{M_+}(-n+m)_{M_-}}{(1)_{|m-s|}}\beta^{|m-s|}(1+\beta^2)^{-n}, \tag{2.9a}$$
$$V_s^{n,m} = F(-n-m+M_+, -n+m+M_-; 1+M_+; \beta^2), \tag{2.9b}$$

with $M_+ = \max(0, m-s)$ and $M_- = \max(0, s-m)$.

In the rest of this paper, we assume that $m - s = M_+ \geqslant 0$ without loss of generality. For $m - s \leqslant 0$, it will be necessary to change the sign of m by $-m$ and s by $-s$ in the established relations (Z-indexes are not concerned by virtue of the symmetry (2.5)).

It follows that we have always $M_- = 0$, $(-n+m)_{M_-} = 1$, and (2.9b) takes the form

$$A_s^{n,m} = \frac{(-n-m)_{m-s}}{(1)_{m-s}}\beta^{m-s}(1+\beta^2)^{-n}, \tag{2.10a}$$
$$V_s^{n,m} = F(-n-m+M_+, -n+m; 1+M_+; \beta^2). \tag{2.10b}$$