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General relativistic radiation hydrodynamics: Mixed-frame approach

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Abstract

I show how general relativistic 3D radiation hydrodynamic equations can be derived from the tensor formulation. Radiation quantities are differentiated with respect to the fixed coordinates while the interaction between matter and radiation is described by the comoving frame quantities. The formulation is covariant, and can be applied to any coordinates or spacetime; I show the derivation for the Schwarzschild spacetime as an example.

Keywords: Accretion; Hydrodynamics; Radiative transfer; Relativity

1. Introduction

High-energy astrophysical systems like X-ray binaries often involve relativistically moving matter and intense radiation field under strong gravity. Such environment demands relativistic treatment of radiation hydrodynamics. Observers moving with relativisitic velocity see photons blue- or redshifted and beamed. Time runs at a different rate for moving observers. Scattering by converging or diverging flow induces bulk Comptonization. When gravity is strong in addition, photons experience gravitational redshift, time ticks differently for observers located at different positions, and the trajectories of photons are bent due to the spacetime curvature, sometimes resulting in a loss-cone effect. All these and other relativistic effects are rather cumbersome to implement one by one into the non-relativistic radiation hydrodynamics. It is much more desirable to build the whole radiation hydrodynamics upon the fully relativistic framework so that all relativistic effects are taken into account naturally.

Thomas (1930) was the first to formulate the special relativistic theory to describe the radiative viscosity in differentially moving media. Lindquist (1966) further derived the covariant moment equations under curved spacetime, but only for a spherically symmetric case. Thorne (1981) used a projected symmetric trace-free tensor formalism to derive the radiation moment equations up to arbitrary order. Thorne's formalism was constructed in the comoving frame. Park (1993), on the other hand, derived mixed-frame radiation hydrodynamic equations under spherical symmetry, in which the interaction of matter and radiation is described by comoving physical quantities while the temporal and spatial derivatives are expressed in the fixed coordinates. This mixed-frame approach makes the equations easier to understand and apply to various problems.

In this talk, I will extend the mixed-frame approach to three-dimensional radiation hydrodynamics in a fully relativisic manner. I will explicitly show how the hydrodynamic and radiation moment equations are derived from covariant energy and momentum conservation for a given spacetime metric, e.g., Schwarzschild metric.

2. Tensor equations

The appropriate physical quantities to describe matter and radiation in covariant manner are the energy-momentum tensor for matter and the radiation stress tensor for radiation, respectively. The energy-momentum tensor of matter that can be approximated by the ideal gas is

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$$T^{\alpha\beta} \equiv \omega_g U^\alpha U^\beta + P_g g^{\alpha\beta}, \tag{1}$$

where U^α is the four velocity of the gas and $\omega_g \equiv \varepsilon_g + P_g$ the gas enthalpy per unit proper volume which is the sum of the gas energy density ε_g and the gas pressure P_g. The radiation stress tensor is

$$R^{\alpha\beta} = \int\int I(\mathbf{n}, \nu) n^\alpha n^\beta \, \mathrm{d}\nu \, \mathrm{d}\Omega, \tag{2}$$

where $I(x^\alpha; \mathbf{n}, \nu)$ is the specific intensity of photons moving in direction $\mathbf{n}$ on the unit sphere of the projected tangent space with the frequency ν measured by the fiducial observer, with $n^\alpha \equiv p^\alpha / h\nu$ and p^α being the four-momentum of photons. Since the combined quantities $\nu^{-3} I_\nu$ and $\nu d\nu d\Omega$ are frame-independent scalars, $R^{\alpha\beta}$ is a contravariant tensor.

The continuity equation in relativistic hydrodynamics becomes the particle number density conservation

$$(nU^\alpha)_{;\alpha} = 0 \tag{3}$$

rather than the mass density conservation because the mass density in relativity includes the internal energy, and is not conserved in general.

In the absence of any external force other than radiative interactions, the total energy and momentum of gas plus radiation is conserved,

$$(T^{\alpha\beta} + R^{\alpha\beta})_{;\beta} = 0. \tag{4}$$

When the micro-physical processes for the interaction between radiation and matter are known, Eq. (4) can be put into two separate equations:

$$T^{\alpha\beta}{}_{;\beta} = G^\alpha \tag{5}$$

for gas and

$$R^{\alpha\beta}{}_{;\beta} = -G^\alpha \tag{6}$$

for radiation, where G^α is the radiation four-force density that represents the energy and momentum transferred from radiation to gas. It is defined as

$$G^\alpha \equiv \frac{1}{c} \int \mathrm{d}\nu \int \mathrm{d}\Omega [\chi I(\mathbf{n}, \nu) - \eta] n^\alpha, \tag{7}$$

where χ is the opacity per unit proper length and η the emissivity per unit proper volume (Mihalas and Mihalas, 1984). The combinations η/ν^2 and $\nu\chi$ are again frame-independent scalars.

3. Schwarzschild spacetime

Although the current formalism can be applied to any spacetime or coordinates, in this talk I choose the Schwarzschild spacetime to explicitly show how the relevant equations are derived. I use the familiar form of the Schwarzschild metric, $\mathrm{d}\tau^2 = \Gamma^2 \mathrm{d}t^2 - \Gamma^{-2} \mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2)$, where M is the mass of the central object, $m \equiv GM/c^2$, and $\Gamma \equiv (1 - 2m/r)^{1/2}$ the lapse function. The four-velocity of the gas $U^\alpha \equiv \mathrm{d}x^\alpha / \mathrm{d}\tau$ satisfies the normalization condition $U_\alpha U^\alpha = -1$, from which the energy parameter $y \equiv -U_t = [\Gamma^2 + (U^r)^2 + \Gamma^2(rU^\theta)^2 + \Gamma^2(r\sin\theta U^\phi)^2]^{1/2}$ is defined.

Physical quantities that can be defined in flat spacetime can be defined similarly in the tetrad frame because it is a locally inertial frame. Fixed and comoving tetrads are generally the most relevant tetrads in radiation hydrodynamics. The fixed tetrad is an orthonormal tetrad fixed with respect to the coordinates and has a base

$$\frac{\partial}{\partial \hat{t}} = \frac{1}{\Gamma}\frac{\partial}{\partial t}, \quad \frac{\partial}{\partial \hat{r}} = \Gamma \frac{\partial}{\partial r}, \quad \frac{\partial}{\partial \hat{\theta}} = \frac{1}{r}\frac{\partial}{\partial \theta}, \quad \frac{\partial}{\partial \hat{\phi}} = \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}. \tag{8}$$

Physical quantities defined in the fixed tetrad are those measured by a fiducial observer who is fixed with respect to the coordinates. This observer sees matter moving with the proper velocity $\mathbf{v}$ with components $v^r = y^{-1}U^r$, $v^\theta = y^{-1}\Gamma r U^\theta$, $v^\phi = y^{-1}\Gamma r \sin\theta U^\phi$. Since $\mathbf{v}$ is a three vector defined in the tetrad, $v_i = v^i$. I also define the Lorentz factor $\gamma \equiv (1 - v^2)^{-1/2} = y \quad \Gamma^{-1}$, where $v^2 = \mathbf{v} \cdot \mathbf{v} = v_i v^i = v_r^2 + v_\theta^2 + v_\phi^2$. The value of v^2 at the horizon is always 1 regardless of U^i: a fiducial observer fixed at the horizon always sees matter radially falling in with velocity c.

The comoving tetrad moves with the velocity v^i relative to the fixed tetrad and therefore is related to the fixed tetrad by the Lorentz tranformation

$$\frac{\partial}{\partial x^{\hat{\alpha}}_{\mathrm{co}}} = \Lambda^{\hat{\beta}}{}_{\hat{\alpha}}(\mathbf{v}) \frac{\partial}{\partial x^{\hat{\beta}}}, \tag{9}$$

where $\hat{\alpha}$ and $\hat{\beta}$ denote each tetrad base. The comoving tetrad, then, can be expressed in terms of coordinate base as

$$\frac{\partial}{\partial \hat{t}_{\mathrm{co}}} = \frac{\gamma}{\Gamma}\frac{\partial}{\partial t} + \gamma \Gamma v_r \frac{\partial}{\partial r} + \gamma v_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \gamma v_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}, \tag{10}$$

$$\frac{\partial}{\partial \hat{r}_{\mathrm{co}}} = \frac{\gamma}{\Gamma} v_r \frac{\partial}{\partial t} + \Gamma \left[1 + (\gamma - 1)\frac{v_r^2}{v^2}\right] \frac{\partial}{\partial r} + (\gamma - 1)\frac{v_r v_\theta}{v^2}\frac{1}{r}\frac{\partial}{\partial \theta} + (\gamma - 1)\frac{v_r v_\phi}{v^2}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}, \tag{11}$$

and similarly for $\partial/\partial\hat{\theta}_{\mathrm{co}}$ and $\partial/\partial\hat{\phi}_{\mathrm{co}}$. Applying the inverse Lorentz transformation $\Lambda^{\hat{\alpha}}{}_{\hat{\beta}}(-\mathbf{v})$ yields the transformation from the comoving tetrad to the coordinate base. These representations are useful when converting tetrad components to and from covariant ones.

4. Radiation moments and four-force density

Although radiation moments of all order are in principle needed to describe arbitrarily anisotropic radiation field, most radiation hydrodynamics deals with moments up to order two, i.e., the zeroth, first, and second moments. The radiation energy density (the zeroth moment), flux vector (the first moments), and pressure tensor (the second moments) in the fixed tetrad are defined as

$$E = \int\int I_\nu \mathrm{d}\nu \mathrm{d}\Omega, \; F^i = \int\int I_\nu n^i \mathrm{d}\nu \mathrm{d}\Omega,$$

$$P^{ij} = \int\int I_\nu n^i n^j \mathrm{d}\nu \mathrm{d}\Omega, \tag{12}$$