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Extending the Li&Ma method to include PSF information

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ABSTRACT

The so called Li&Ma formula is still the most frequently used method for estimating the significance of observations carried out by Imaging Atmospheric Cherenkov Telescopes. In this work a straightforward extension of the method for point sources that profits from the good imaging capabilities of current instruments is proposed. It is based on a likelihood ratio under the assumption of a well-known PSF and a smooth background. Its performance is tested with Monte Carlo simulations based on real observations and its sensitivity is compared to standard methods which do not incorporate PSF information. The gain of significance that can be attributed to the inclusion of the PSF is around 10% and can be boosted if a background model is assumed or a finer binning is used.

1. Introduction

The statistical significance of an observation is a key issue in *signal starved* fields such as Imaging Atmospheric Cherenkov Telescopes (IACTs) Astronomy, and in general Very High Energy (VHE) Astronomy. It determines whether a given astronomical source has been detected or not, providing a probability for the excess being due to background fluctuations. It also limits how much detail can be recovered in spectra and light curves, because a minimum significance is usually required for each spectral or light curve point to be accepted. Finally, it also plays an important role when the goal is to set upper limits for non-detected sources. In this case, the sensitivity of the method determines how constraining the upper limit is.

Until the publication of the classical article by Li&Ma [1], several approaches to define the significance of astronomical observations had been used in VHE observations. As shown in that article, most of them were based on incorrect statistical hypotheses, and thus yielded unexpected widths of the significance distributions when they were tested with Monte Carlo (MC) simulations. In their article, Li&Ma proposed a robust and reliable method for estimating that significance. Since at that time VHE instrumentation had very limited angular resolution, the method was designed as an event counting technique which makes very little use of the instrument resolution, given by its Point Spread Function (PSF), and background distribution. Therefore, the sensitivity achieved should be worse than the one of methods that incorporate that information.

The *Li&Ma* method, which shall be known as just *Li&Ma* in the rest of this work, is a particular case of a more general family of techniques based on maximum likelihood principles. Generalized maximum likelihood methods such as that implemented in [2,3] are sometimes difficult to implement. There have been general proposals such as [4] to extend the *Li&Ma* formula or include the effect of systematic errors (e.g. [5,6]). Still, the use of general likelihood methods in IACT Astronomy is restricted in practice to special analyses such as sky maps [7] or spectral line studies in Dark Matter searches, as seen in [8,9]. Nevertheless, even if they risk losing robustness and stability, they are usually more sensitive than event counting methods.

In this article, a simple technique that takes into account the a priori knowledge of the instrumental PSF is presented and characterized in detail, under the assumption of a smooth background for which dedicated measures are available. Although the method is applicable to a wide range of situations, it has been tested in our field of interest: VHE observations. It can be understood as a generalization of the *Li&Ma* method or a particular application of that proposed in [4] to a specially relevant case: the search for one isolated point source in the field of view (FoV), which is the common case in extragalactic observations with the current sensitivity of IACT experiments. A point source is defined as one whose angular size is smaller than the PSF of the instrument. Known as the *PSF-Likelihood* method it can recover more information from the source of interest while keeping, at the same time, the simplicity of the standard *Li&Ma* method. In order to check whether the statistical foundations of the technique are correct, and estimate its rejection power, it is tested with a set of *toy Monte Carlo* samples generated using real background and data from observations of the Crab Nebula performed by the MAGIC experiment [10]. The comparison allows what can be gained from this kind of approach in a real situation to be gauged.

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1.1. Maximum likelihood with background estimation

IACTs operate in harsh environments and their performance is highly dependent on the atmospheric and instrumental observing conditions. As a consequence the background affecting an observation is highly variable and is usually estimated jointly with the signal. In the past, the observation time was divided between ON observations, in which the telescope was pointed towards the source, and OFF observations, in which the telescope was pointed to an equivalent region with no source present. Nowadays it is customary to use alternative methods that do not require dedicated OFF observations. This is the case of the *Wobble* method, in which the telescope is pointed to different positions at a small fixed distance from the source. The size of the IACTs Field of View (FoV) makes it possible to take simultaneous ON and OFF data, as described in [10–12]. Sometimes it is possible to define several OFF regions within the same field, but additional care should be taken to avoid counting events twice.

All the significance estimators tested in this article are based on a binned Maximum Likelihood Ratio approach, which tests an assumed null hypothesis against an alternative one, formulated as:

Null hypothesis, H0 ON and OFF regions contain no sources, only background.

Alternative hypothesis, H1 While the OFF region only contains background, in the ON region there is, in addition, a source.

A simple case, in which the result of an observation is a one-dimensional histogram showing the number of events detected as a function of the squared distance to the source, will be assumed. The number of events per bin will follow Poisson statistics, leading to the Likelihood function:

$$\mathcal{L}(X|\Theta) = \prod_{i}^{N} \frac{f_i^{n_i}(\Theta)e^{-f_i(\Theta)}}{n_i!} \tag{1}$$

where Θ is the parameter space for our model, i is the bin index (for a total of N bins), n_i the number of events in bin i and f_i the value of the test model in the given bin.

It is often convenient to work with the negative logarithm of this function,

$$\begin{aligned}\mathbf{L}(X|\Theta) &\equiv -\log \mathcal{L}(X|\Theta) \\ &= -\sum_{i}^{N} n_i \log f_i(\Theta) - f_i(\Theta) - \log n_i! \end{aligned} \tag{2}$$

Since the last term of the summation is only a normalization factor, which does not depend on the parameters of the likelihood (Θ), it can be safely removed and the expression simplified to:

$$\mathbf{L}'(X|\Theta) = -\sum_{i}^{N} n_i \log f_i(\Theta) - f_i(\Theta) \tag{3}$$

Now the likelihood ratio λ and its logarithm can be computed, giving:

$$\begin{aligned}-2\log\lambda &\equiv -2\log\left[\frac{\mathcal{L}_{H0}(X|\Theta)}{\mathcal{L}_{H1}(X|\Theta)}\right] \\ &= 2\{\mathbf{L}'_{H0}(X|\Theta) - \mathbf{L}'_{H1}(X|\Theta)\}\end{aligned} \tag{4}$$

From [13] it is known that, when the null hypothesis is true, $-2\log\lambda$ asymptotically follows a χ_r^2 distribution for large event counts, where r is the difference in the number of degrees of freedom between both hypotheses. This can be used to compute the probability of the observed excess being due to a background fluctuation. It can also be translated into a test statistics $TS = \chi_1^2$, where χ_1^2 is the value of χ^2 with one degree of freedom corresponding to the same probability as the original χ_r^2. The accurate approximation proposed by Wallace [14] can be used to compute the corresponding value in the limit of high *TS*, while its sign can be set from the sign of the event excess.

Table 1
Expected improvements from Li&Ma Formula 17 with the number of OFF positions ($1/\alpha$) for the particular case $n_{off} = 640/\alpha$ and $n_{on} = 160 + 640$.

	Number of OFF positions					
	1	3	5	9	15	∞
$S(\sigma)$	4.2	5.2 +24%	5.5 +31%	5.8 +36%	5.9 +39%	6.1 +44%

1.1.1. The Li&Ma method

In the *Li&Ma* method, where $r = 1$, only one bin is defined in each, ON and OFF regions. Then *TS* has an analytical expression, which is normally known as the Li&Ma formula (see [1, Formula 17]). It depends on n_{on} and n_{off}, the number of ON and OFF events respectively and α, the ratio between the effective ON and OFF observation times. A source region must be selected *a priori* to count ON and OFF events, which must be done carefully to avoid losing sensitivity. It is usually chosen as the one giving the maximum significance in a test sample (typically a Crab Nebula test sample) taking into account the PSF of the instrument and the expected background statistics.

1.1.2. The Li&Ma with fit background method

The number of OFF events can also be obtained by incorporating information from a region larger than that considered in *Li&Ma*, by fitting a background model against the data and integrating the model in the selected signal region. This method usually gives smaller statistical uncertainties, as it is in principle equivalent to having better OFF statistics. In order to use this model, one must be aware of any existing inhomogeneity in the camera or other gamma-ray sources which would introduce additional components in the background shape. An additional constrain exists if Wobble-mode observations are performed, as the wobble offset (distance between the source position and the actual pointing position) limits the maximum range of the fit that can be used without double-counting events.

We will call this variant hereafter the *Li&Ma with fit background* method. It can be implemented by calculating modified $\alpha' \equiv \alpha\sqrt{\frac{(\delta n_{off})^2}{n_{off}}}$ and $n'_{off} \equiv n_{off}\frac{\alpha}{\alpha'}$ values, where δn_{off} is the estimated n_{off} uncertainty. In this case, δn_{off} is no longer the Poisson based $\sqrt{n_{off}}$, but the total uncertainty estimated using the fit covariance matrix. α is the actual ratio between the effective ON and OFF time. These new α' and n'_{off} values can be inserted into the *Li&Ma* formula to get the significance.

1.1.3. Other background estimation methods

There are other ways of increasing the effective statistics in the background region and thus to potentially improve the sensitivity. One clear example is to increase the number of OFF regions as discussed in Section 2.3 of [15]. An example of the gain that can be obtained with this approach is shown in Table 1. The main advantage of this method is that all the positions remain symmetric with respect to the center of the camera and the relative radial response is the same as in the ON region, which means that the only assumptions that are needed are a radially symmetric camera response, no significant sky changes among the different OFF regions and no additional sources present in the selected OFF positions. These requirements are different from those required in *Li&Ma with fit background* formula and the best solution would thus depend on the particularities of the given instrument.

Another example is the so called *Ring method* [15]. The main advantages of this method is that its symmetry properties make it less prone to systematic errors due to sky gradients. In principle it can be applied to any point of the FoV and it is conceptually similar to other aperture photometry methods widely used in Astronomy.