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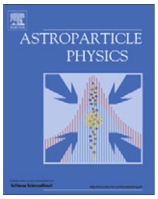

Leading relativistic corrections to the Kompaneets equation

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ABSTRACT

We calculate the first relativistic corrections to the Kompaneets equation for the evolution of the photon frequency distribution brought about by Compton scattering. The Lorentz invariant Boltzmann equation for electron–photon scattering is first specialized to isotropic electron and photon distributions, the squared scattering amplitude and the energy–momentum conserving delta function are each expanded to order v^4/c^4, averages over the directions of the electron and photon momenta are then carried out, and finally an integration over the photon energy yields our Fokker–Planck equation. The Kompaneets equation, which involves only first- and second-order derivatives with respect to the photon energy, results from the order v^2/c^2 terms, while the first relativistic corrections of order v^4/c^4 introduce third- and fourth-order derivatives. We emphasize that our result holds when neither the electrons nor the photons are in thermal equilibrium; two effective temperatures characterize a general, non-thermal electron distribution. When the electrons are in thermal equilibrium our relativistic Fokker–Planck equation is in complete agreement with the most recent published results, but we both disagree with older work.

1. Introduction

The Kompaneets [1] equation,

$$\frac{\partial}{\partial t}f(t,\omega) = \frac{\sigma_{\mathrm{T}} n_e}{m_e c}\frac{1}{\omega^2}\frac{\partial}{\partial\omega}\omega^4\left\{T\frac{\partial f(t,\omega)}{\partial\omega} + \hbar[1+f(t,\omega)]f(t,\omega)\right\}, \tag{1.1}$$

describes the scattering of unpolarized, low energy photons of frequency ω on a dilute distribution of non-relativistic electrons when all the particles — photons and electrons — are distributed isotropically in their momenta. The non-relativistic total photon–electron cross section is the Thomson cross section σ_{T}. The electron number density and mass are denoted by n_e and m_e. The photon phase space distribution $f(t,\omega)$ is normalized such that the number n_γ of photons per unit volume is given by

$$n_\gamma(t) = 2\int\frac{(d^3\mathbf{k})}{(2\pi)^3}f(t,\omega), \tag{1.2}$$

in which the prefactor 2 counts the number of photon polarization states and $\mathbf{k}$ is the photon wave-number vector with $|\mathbf{k}|c = \omega$. If the electrons are in thermal equilibrium described by a Maxwell–Boltzmann distribution, then T is the temperature (in energy units) T_e of this thermal distribution. However, the Kompaneets equation (1.1) holds for any isotropic distribution of electron momenta with T defined to be 2/3 of the average energy in this distribution [2]. For photons with a Planck distribution,

$$f(t,\omega) \to f^{(0)}(\omega) = \frac{1}{\exp\{\hbar\omega/T_\gamma\} - 1}. \tag{1.3}$$

The terms in the curly braces in the Kompaneets equation (1.1) vanish when $T_\gamma = T$. In particular, if $T = T_\gamma = T_e$, there is a time-independent photon distribution in thermal equilibrium with the electrons.[1]

Our purpose here is to examine the first relativistic corrections to the Kompaneets equation. These corrections have been previously computed by Challinor and Lasenby (C&L) [3] for the case in which the electrons are in a thermal distribution. Using the method of C&L, Itoh et al. [4] carried out the expansion to a much higher order in v/c. Subsequently, Sazonov and Sunyaev [5] confirmed the previous work of Challinor and Lasenby. Here we use a method that is quite different from that employed by C&L, a method that does not require that the electrons be in thermal equilibrium. Moreover, this method explicitly exhibits the order of v/c in every term and thus provides a straightforward evaluation of the correct v^2/c^2 corrections to the Kompaneets equation. Although the structure of our result is quite different from that found by C&L, we agree with C&L in the number of higher-order derivatives with respect to the photon frequency ω which must supplement the Kompaneets equation to correctly account for the relativistic

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[1] Since Compton scattering preserves the photon number, the collision term on the right-hand side of Eq. (1.1) also vanishes for a general Bose-Einstein distribution of massless particles at temperature T, $f(t,\omega) \to f^{(\alpha)}(\omega) = [\exp\{(\hbar\omega/T) - \alpha\} - 1]^{-1}$.

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corrections. Such higher-order derivative terms are missing from the ad hoc treatments of Cooper [6] and of Prasad et al. [7]. These authors assume (incorrectly) that the relativistic corrections may be accounted for by simply replacing the factor ω^4 that stands just before the curly braces in Eq. (1.1) by a function $\alpha(\omega,T)$ which is determined so as to give the rate of change of the photon energy density including the first relativistic corrections. We compute both the rate of energy exchange between the photons and electrons and the Sunyaev–Zel'dovich effect [9–11] which follow from the relativistically corrected Kompaneets equation. Including the first relativistic corrections, our results entail two effective temperatures $T_{\mathrm{eff}1}$ and $T_{\mathrm{eff}2}$ which are defined by energy moments of the electron phase-space distribution. When the electron distribution is restricted to a thermal, relativistic Maxwell–Boltzmann distribution at temperature T, $T_{\mathrm{eff}1} = T_{\mathrm{eff}2} = T$ and we find, after some algebra, that our results that have a completely different structure are, in fact, in complete agreement with those of C&L. Moreover, the rate of energy exchange that we compute (also written down by C&L) agrees with that found earlier by Woodward [8].

Our presentation is organized as follows: After describing the general method we use in Section 2, we then outline the calculation in Section 3 using the results of several Appendices. Finally, our results are shown in Section 4: Section 4.1 presents our general result, Section 4.2 gives its restriction to the case in which the photons are in thermal equilibrium at temperature T_γ, Section 4.3 derives the rate of energy transport between photons at temperature T_γ and the electrons in a general distribution, and finally, in Section 4.4 the Sunyaev–Zel'dovich effect for non-thermal electrons with the first relativistic correction is briefly described.

2. Relativistic Boltzmann equation for isotropic scattering

We start from the Lorentz invariant form of the Boltzmann equation for electron–photon scattering:

$$k\partial f(x,k) = \int \frac{(d^3\mathbf{p}')}{(2\pi)^3}\frac{1}{2E'}\frac{(d^3\mathbf{k}')}{(2\pi)^3}\frac{1}{2\omega'}\frac{(d^3\mathbf{p})}{(2\pi)^3}\frac{1}{2E}(2\pi)^4 \times \delta^4(p'+k'-p-k)\left|T(p',k';p,k)\right|^2\{[1+f(x,k)] \times 2g(x,p')f(x,k') - [1+f(x,k')]2g(x,p)f(x,k)\}. \tag{2.1}$$

Here we revert to units in which $\hbar = 1 = c$, but we shall return to conventional units when we write the final result. The left-hand side of the equation involves the relativistic scalar $k\partial = \omega(\partial/\partial t) + \mathbf{k}\cdot\nabla$. We are assuming that the electrons and photons are not polarized. Hence $|T|^2$ denotes the square of the Lorentz invariant scattering amplitude that is summed over the initial and final electron and photon spins. It is divided by the initial electron spin weight $g_e = 2$ so as to describe the average scattering from an initially unpolarized ensemble of electrons. It is divided by the square of the photon spin weight $g_\gamma^2 = 4$ because initially there is an unpolarized mixture and finally the scattering is into the scalar density $f(x,k)$ that describes a typical photon (with the factor $g_\gamma = 2$ needed to provide the photon number count in Eq. (1.2)). The function $g(x,p)$ is the electron phase space density. We choose our Lorentz metric to have signature $(-{+}{+}{+})$ so that $t = x^0 = -x_0$ while for the spatial coordinates $x^k = x_k$.

We now specialize to the isotropic case of interest where $f(x,k) \to f(t,\omega)$ and $g(x,p) \to g(t,E)$, with the electron number density given by

$$n_e = 2\int \frac{(d^3\mathbf{p})}{(2\pi)^3} g(t,E). \tag{2.2}$$

The integration variables p and p' in Eq. (2.1) are dummy variables. We shall make the interchange $p \leftrightarrow p'$ in the first set of terms in Eq. (2.1) so as to have a common factor of $g(t,E)$ for the two 'scattering in to' and 'scattering out of' terms. To keep a convenient form, we shall also use the detailed balance relation

$$\left|T(p',k';p,k)\right|^2 = \left|T(p,k;p',k')\right|^2 \tag{2.3}$$

for this first term in Eq. (2.1). Finally, we note that the $\mathbf{p}'$ integration is best performed using

$$\frac{(d^3\mathbf{p}')}{(2\pi)^3}\frac{1}{2E'} = \frac{(d^3\mathbf{p}')}{(2\pi)^3}\frac{1}{2\sqrt{\mathbf{p}'^2+m_e^2}} = \frac{(d^4p')}{(2\pi)^3}\delta\left(p'^2+m_e^2\right) \tag{2.4}$$

against the four-dimensional delta function which now replaces

$$p' = p + k - k' \tag{2.5}$$

giving

$$p'^2 + m_e^2 = 2p(k-k') - 2kk'. \tag{2.6}$$

In this fashion, we obtain

$$\omega\frac{\partial}{\partial t}f(t,\omega) = \int \frac{(d^3\mathbf{k}')}{(2\pi)^3}\frac{1}{2\omega'}\frac{(d^3\mathbf{p})}{(2\pi)^3}\frac{1}{2E}2g(t,E)2\pi\delta(2p(k-k') - 2kk')|T(p',k;p,k')|^2\{[1+f(t,\omega)]f(t,\omega') - |T(p',k';p,k)|^2[1+f(t,\omega')]f(t,\omega)\Big\}, \tag{2.7}$$

in which the four-momentum p' in $|T(p',k;p,k')|^2$ is determined by Eq. (2.5).

The angular part of the integrations over $\mathbf{p}$ and $\mathbf{k}'$ pick out the completely rotationally invariant part of the integrand. Thus, with angular brackets denoting the average over all the orientations of the vectors within it, we may make the replacement

$$\delta(2p(k-k') - 2kk')|T(p',k';p,k)|^2 \to \left\langle \delta(2p(k-k') - 2kk')|T(p',k';p,k)|^2 \right\rangle \equiv s(p;\omega',\omega). \tag{2.8}$$

In view of these remarks, we may write Eq. (2.7) as

$$\frac{\partial}{\partial t}f(t,\omega) = \frac{1}{\omega^2}\int \frac{(d^3\mathbf{p})}{(2\pi)^3}\frac{1}{2E}2g(p)F(t,\omega;p) \tag{2.9}$$

with

$$F(t,\omega;p) = \frac{\omega}{2\pi}\int_0^\infty \omega' d\omega'\{s(p;\omega,\omega')[1+f(t,\omega)]f(t,\omega') - s(p;\omega',\omega)[1+f(t,\omega')]f(t,\omega)\}. \tag{2.10}$$

For the evaluation of Eq. (2.10) it is convenient to separate the weight that appears there into symmetric and antisymmetric parts:

$$s(p;\omega',\omega) = s^{\mathrm{S}}(p;\omega',\omega) + s^{\mathrm{A}}(p;\omega',\omega) \tag{2.11}$$

with

$$s^{\mathrm{S}}(p;\omega,\omega') = +s^{\mathrm{S}}(p;\omega',\omega) \tag{2.12}$$

and

$$s^{\mathrm{A}}(p;\omega,\omega') = -s^{\mathrm{A}}(p;\omega',\omega). \tag{2.13}$$

With this decomposition, Eq. (2.10) becomes

$$F(t,\omega;p) = \frac{\omega}{2\pi}\int_0^\infty \omega' d\omega'\left[s^{\mathrm{S}}(p;\omega',\omega) - s^{\mathrm{A}}(p;\omega',\omega)\right]f(t,\omega') - \frac{\omega}{2\pi}f(t,\omega)\int_0^\infty \omega' d\omega'\left[s^{\mathrm{S}}(p;\omega',\omega) + s^{\mathrm{A}}(p;\omega',\omega)\right] - \frac{\omega}{2\pi}2f(t,\omega)\int_0^\infty \omega'\, d\omega' s^{\mathrm{A}}(p;\omega',\omega)f(t,\omega'). \tag{2.14}$$