



## Note

# Tidal dissipation at arbitrary eccentricity and obliquity

Jack Wisdom

*Massachusetts Institute of Technology, 54-414 MIT, Cambridge, MA 02139, USA*

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## Abstract

Expressions for tidal dissipation in a body in synchronous rotation at arbitrary orbital eccentricity and obliquity are derived. The rate of tidal dissipation for a synchronously rotating body is compared to that in a body in asymptotic nonsynchronous rotation.

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## 1. Introduction

Time-dependent tidal distortion of a body leads to internal heating. For a synchronously rotating body in an eccentric orbit the rate of energy dissipation is reported to be (Peale and Cassen, 1978; Peale et al., 1979; Wisdom, 2004)

$$\frac{dE}{dt} = \frac{21}{2} \frac{k_2}{Q} \frac{GM^2 n R^5 e^2}{a^6}, \quad (1)$$

where  $k_2$  is the satellite (secondary body) potential Love number,  $Q$  is the satellite effective tidal dissipation parameter,  $G$  is the gravitational constant,  $a$  is the orbit semimajor axis,  $M$  the mass of the host planet (primary body),  $n$  the orbital mean motion (and rotation rate), which is approximately  $\sqrt{GM/a^3}$ ,  $R$  is the satellite radius, and  $e$  is the orbital eccentricity. The derivation of this formula assumes that the body is incompressible, the rotation is uniform and synchronous, and that the body is small enough that the displacement Love number  $h_2$  is  $5k_2/3$ . The eccentricity has also been assumed to be small, and only the lowest order factor in eccentricity has been kept. Wisdom (2004) generalized this expression to include the lowest order terms in obliquity, forced synchronous libration, and spin–orbit secondary libration.

Though this expression has been adequate for most discussions of tidal dissipation in the Solar System where the orbital eccentricities are relatively small, there are also now situations for which it is inadequate. For instance, Garrick-Bethell et al. (2006) argued that the shape of the Moon was best explained if the moment differences of the Moon froze in during a period in which the Moon had large eccentricity. For synchronous rotation, the orbit that satisfies the shape constraint has an eccentricity of 0.49. We will see that in this case the familiar formula for tidal dissipation, Eq. (1), underestimates the rate of tidal dissipation by a factor of about 30. Another possible situation of interest is tidal dissipation in extrasolar planets. Extrasolar planets have been found to have a wide range of orbital eccentricities. For those extrasolar planets that are gas giants, it may be unlikely that they are in synchronous rotation, but rather may be expected to be in an asymptotic non-

synchronous state as was once proposed for Mercury (Peale and Gold, 1965; Levrard et al., 2007). Nevertheless, it may be expected that more rocky extrasolar planets will be discovered and that some of these may also have large orbital eccentricities and be in synchronous rotation. Extrasolar planets may also have large obliquity: Winn and Holman (2005) proposed that HD209458b might be in a high obliquity Cassini state, and that the enhanced tidal dissipation at large obliquity would inflate its radius (see also the discussion of HD209458b by Levrard et al., 2007).<sup>1</sup> It will be appropriate then to have a valid expression for tidal dissipation at arbitrary eccentricity and obliquity.

Here we generalize the familiar result for tidal dissipation in a synchronously rotating satellite, Eq. (1), and derive a concise formula applicable at arbitrary eccentricity and obliquity. We compare the rate of tidal heating in synchronous rotation to that in asymptotic nonsynchronous rotation (Levrard et al., 2007).

## 2. Derivation for synchronous rotation

The derivation of the generalizations of the tidal heating formula that is presented in Wisdom (2004) follows one of the derivations presented in Peale and Cassen (1978). Here we follow the same derivation, but, as appropriate, generalize the expressions so that they are applicable at arbitrary eccentricity and obliquity. In this section we assume the obliquity is zero.

Following Wisdom (2004),<sup>2</sup> let  $U_T$  be the tide-raising potential. The satellite may be thought of as consisting of a myriad of small constituent mass

<sup>1</sup> In Wisdom (2004), it is clear that the expressions are only valid to second order in eccentricity, but I forgot to state that the results were also truncated at second order in the obliquity. Unfortunately, Winn and Holman (2005) used the expression for tidal dissipation in synchronous rotation presented in Wisdom (2004) at large obliquity. And Levrard et al. (2007) used the expressions in Wisdom (2004) for the synchronous dissipation rate at both high eccentricity and high obliquity.

<sup>2</sup> For ease of reading, some of that derivation is repeated here (and corrected).

E-mail address: [wisdom@mit.edu](mailto:wisdom@mit.edu).

elements. The force on each mass element is the negative gradient of the potential energy, where the potential energy is the tidal potential multiplied by the mass of the constituent. The rate at which work is done on each constituent is the dot product of this force with the velocity of the constituent. Integrating over the volume of the satellite gives the rate at which work is done on the satellite.

The rate of energy dissipation in the satellite is

$$\frac{dE}{dt} = - \int_{\text{Body}} \rho \vec{v} \cdot \nabla U_T dV, \quad (2)$$

where  $\rho$  is the density and  $dV$  is the volume element. To a good approximation a satellite may be assumed to be incompressible  $\nabla \cdot \vec{v} = 0$  (Peale and Cassen, 1978). The product rule gives

$$\nabla \cdot (U_T \vec{v}) = \vec{v} \cdot \nabla U_T + U_T \nabla \cdot \vec{v}, \quad (3)$$

so, with the assumption of incompressibility, the rate of energy dissipation is

$$\frac{dE}{dt} = - \int_{\text{Body}} \rho \nabla \cdot (U_T \vec{v}) dV. \quad (4)$$

If we ignore any variation of density in the body, Gauss's theorem allows us to write the rate of energy dissipation as a surface integral

$$\frac{dE}{dt} = -\rho \int_{\text{Surface}} U_T \vec{v} \cdot \vec{n} dS, \quad (5)$$

where  $\vec{n}$  is the normal to the surface and  $dS$  is the surface area element. Now  $\vec{v} \cdot \vec{n}$  is the rate at which the height of the surface changes. The height of the tide at any point on the surface is approximately

$$\Delta r = -\frac{h_2 U_T'}{g}, \quad (6)$$

where  $h_2$  is the displacement Love number for the satellite,  $g$  is the local acceleration of gravity, and the prime on  $U_T$  indicates that the tidal potential is given a phase delay because the dissipative tidal response lags the forcing. So

$$\frac{dE}{dt} = \frac{\rho h_2}{g} \int_{\text{Surface}} U_T \frac{d}{dt} (U_T') dS. \quad (7)$$

The tide-raising gravity-gradient potential is

$$U_T = -\frac{GM R^2}{r^3} P_2(\cos \alpha), \quad (8)$$

where  $P_2$  is the second Legendre polynomial,  $\alpha$  is the angle at the center of the satellite between the planet to satellite line to the point in the satellite where the potential is being evaluated,  $R$  is the distance from the satellite center to the evaluation point, and  $r$  is the planet to satellite distance.

Consider motion in a fixed elliptical orbit, with eccentricity  $e$ . Choosing rectangular coordinates with the  $x$ -axis aligned with the orbit pericenter and the orbit in the  $(x, y)$ -plane, the orbital position is

$$\mathbf{o} = (r \cos f, r \sin f, 0) \quad (9)$$

with true anomaly  $f$ . In terms of planetocentric longitude  $\lambda$  and colatitude  $\theta$ , the rectangular components of a surface element in the unrotated satellite (or at the initial time) are

$$\mathbf{s}_0 = (R \sin \theta \cos \lambda, R \sin \theta \sin \lambda, R \cos \theta). \quad (10)$$

Assuming uniform synchronous rotation about the  $z$ -axis (perpendicular to the orbit plane), the rectangular components of this element at time  $t$  are

$$\mathbf{s} = \mathbf{R}_z(nt) \mathbf{s}_0 = (R \sin \theta \cos(\lambda + nt), R \sin \theta \sin(\lambda + nt), R \cos \theta), \quad (11)$$

where  $\mathbf{R}_z(nt)$  is an active right-handed rotation about the  $z$ -axis by the angle  $nt$ . The dot product of the surface element with the orbital position gives  $\mathbf{o} \cdot \mathbf{s} = r R \cos \alpha$ . This completes the expression for the tidal potential as a function of time and location on the surface. The delayed tidal potential  $U_T'$  is found by

replacing  $nt$  by  $nt + \Delta$  in the expression for  $U_T$ . The tidal model used here is the one where  $1/Q$  is proportional to frequency. This is sometimes known as the Mignard model (Mignard, 1980).

The average rate of energy dissipation is found by carrying out the surface integral, Eq. (7), and averaging over an orbital period.

The surface integral may be carried out by expanding the integrand as a Poisson series in the angular variables. Any term containing  $\lambda$  then integrates to zero, the rest are multiplied by  $2\pi$ . The  $\theta$  integrals are simple. The details are unilluminating and will not be shown. The calculations were carried out with computer algebra, and checked by performing the integrals numerically. The result is

$$\frac{dE}{dt} = \frac{\rho h_2}{g} \frac{G^2 M^2 R^6 n}{a^6} \gamma, \quad (12)$$

where

$$\begin{aligned} \frac{\gamma}{2\pi} = & \frac{3}{5} \left(\frac{a}{r}\right)^3 \left(\frac{a}{r'}\right)^5 \beta \sin(2\Delta + 2f - 2f') \\ & - \frac{3}{10} \left(\frac{a}{r}\right)^3 \left(\frac{a}{r'}\right)^4 \frac{e}{\beta} \sin(f') \\ & - \frac{9}{20} \left(\frac{a}{r}\right)^3 \left(\frac{a}{r'}\right)^4 \frac{e}{\beta} \sin(2\Delta + 2f - f') \\ & + \frac{9}{20} \left(\frac{a}{r}\right)^3 \left(\frac{a}{r'}\right)^4 \frac{e}{\beta} \sin(2\Delta + 2f - 3f') \\ & - \frac{3}{5} \left(\frac{a}{r}\right)^3 \left(\frac{a}{r'}\right)^3 \sin(2\Delta + 2f - 2f') \end{aligned} \quad (13)$$

and where  $\beta = (1 - e^2)^{1/2}$ . The values of  $r'$  and  $f'$  are those of the radius  $r$  and true anomaly  $f$  for a mean anomaly of  $nt + \Delta$ .

We assume the dissipation is small and can therefore approximate

$$\gamma = \gamma_0 + \frac{d\gamma}{d\Delta} \Delta + \dots \approx \gamma_0 + \frac{d\gamma}{d\Delta} \sin \Delta, \quad (14)$$

where  $\gamma_0$  is the value of  $\gamma$  for  $\Delta = 0$ , and the derivative  $d\gamma/d\Delta$  is evaluated at  $\Delta = 0$ . With this approximation the time average of the energy dissipation expression can be completed analytically.

Using

$$\frac{df}{d\mathcal{M}} = \beta \left(\frac{a}{r}\right)^2 \quad (15)$$

and

$$\frac{d(a/r)}{d\mathcal{M}} = -\frac{e \sin f}{\beta^2} \frac{df}{d\mathcal{M}}, \quad (16)$$

where  $\mathcal{M}$  is the mean anomaly, we find

$$\begin{aligned} \frac{1}{2\pi} \frac{d\gamma}{d\Delta} = & -\frac{6}{5} \left(\frac{a}{r}\right)^{10} \beta^2 - \frac{6}{5} \left(\frac{a}{r}\right)^9 e \cos f - \frac{6}{5} \left(\frac{a}{r}\right)^6 \\ & + \frac{12}{5} \left(\frac{a}{r}\right)^8 \left[ \beta - \frac{e^2}{\beta^2} (1 - \cos(2f)) \right]. \end{aligned} \quad (17)$$

The time average of the energy dissipation can be found by integrating the energy dissipation over an orbital period and dividing by the orbital period. Note that  $\gamma_0$  is proportional to  $\sin f/r^7$ , and its time average is zero. Thus we just need to calculate the average of  $d\gamma/d\Delta$ .

The integrals involving the radius and true anomaly can be expressed exactly in terms of Hansen functions  $X_k^{ij}$  (Plummer, 1960; Mignard, 1980). The time average of the energy dissipation is

$$\frac{dE}{dt} = \frac{\rho h_2}{g} \frac{G^2 M^2 R^6 n}{a^6} \left\langle \frac{d\gamma}{d\Delta} \right\rangle \sin \Delta, \quad (18)$$

where  $\langle d\gamma/d\Delta \rangle$  is the average of  $d\gamma/d\Delta$  over an orbit period:

$$\begin{aligned} \frac{1}{2\pi} \left\langle \frac{d\gamma}{d\Delta} \right\rangle = & -\frac{6}{5} X_0^{-10,0}(e) \beta^2 - \frac{6}{5} X_0^{-9,1}(e) e - \frac{6}{5} X_0^{-6,0}(e) \\ & + \frac{12}{5} \left[ X_0^{-8,0}(e) \left( \beta - \frac{e^2}{\beta^2} \right) + \frac{e^2}{\beta^2} X_0^{-8,2}(e) \right]. \end{aligned} \quad (19)$$

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