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Electron acceleration with improved Stochastic Differential Equation method: Cutoff shape of electron distribution in test-particle limit

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ABSTRACT

We develop a method of stochastic differential equation to simulate electron acceleration at astrophysical shocks. Our method is based on Itô's stochastic differential equations coupled with a particle splitting, employing a skew Brownian motion where an asymmetric shock crossing probability is considered. Using this code, we perform simulations of electron acceleration at stationary plane parallel shock with various parameter sets, and studied how the cutoff shape, which is characterized by cutoff shape parameter a, changes with the momentum dependence of the diffusion coefficient β. In the age-limited cases, we reproduce previous results of other authors, $a \approx 2\beta$. In the cooling-limited cases, the analytical expectation $a \approx \beta + 1$ is roughly reproduced although we recognize deviations to some extent. In the case of escape-limited acceleration, numerical result fits analytical stationary solution well, but deviates from the previous asymptotic analytical formula $a \approx \beta$.

1. Introduction

Mechanism of particle acceleration is still unknown. Diffusive shock acceleration (Krymskii, 1977; Bell, 1978; Blandford and Ostriker, 1978) is the most plausible if strong shock waves exist as in young supernova remnants (SNRs). We have not yet well constrained model parameters, namely magnetic field strength and degree of magnetohydrodynamic turbulence, although there are observational claims of turbulent, amplified field in young SNRs (Vink and Laming, 2003; Bamba et al., 2003, 2005a, 2005b; Yamazaki et al., 2004; Uchiyama et al., 2007). These are important to estimate maximum attainable energy of both electrons and nuclei (e.g., Yoshida and Yanagita, 1997). Yamazaki et al. (2013) proposed that cutoff shape of electron spectrum around the maximum energy $E_{\rm max}$ may provide us important information on the cosmic-ray acceleration at young SNRs. They related the cutoff shape parameter a, which is defined by $N(E) \propto \exp[-(E/E_{\rm max})^a]$, to the energy dependence of the electron diffusion coefficient β (that is, $K \propto E^\beta$) in each case where the maximum electron energy is determined by SNR age, synchrotron cooling and escape from the shock. They found that if the power-law index of the electron spectrum is independently determined by other observations, then the cutoff shape parameter can be constrained by near future hard X-ray observations such as Nuclear Spectroscopic Telescope Array (NuSTAR) (Hailey et al., 2010; Harrison et al., 2013) and ASTRO-H (Takahashi et al., 2010) and/or CTA (Actis et al., 2011). These X-ray and gamma-ray observations will be important for the estimate of β as well as $E_{\rm max}$ and the magnetic field strength.

In analysis of Yamazaki et al. (2013), they assumed relations between a and β as $a = 2\beta$, $\beta + 1$ and β in the case of age-limited, cooling-limited and escape-limited acceleration, respectively. The formula $a = 2\beta$ in the age-limited case has been based on numerical simulation (Kato and Takahara, 2003; Kang et al., 2009), while the others are obtained analytically on the assumption of stationary state, and they are not yet confirmed numerically. In this paper, we study the cutoff shape of the electron spectrum by numerically solving the transport equation describing diffusive shock acceleration, and study whether the above relations are right or not.

We use a numerical method for solving cosmic-ray transport equation (so-called, diffusion–convection equation), which was proposed by Achterberg and Krülls (1992). This method is based on the equivalence between the Fokker–Planck equation and the Itô stochastic differential equation (SDE) (Gardiner, 1983). Subsequent studies have followed for various situations (Krülls and Achterberg, 1994; Yoshida and Yanagita, 1994; Marcowith and Kirk, 1999; Marcowith and Casse, 2010; Schure et al., 2010). It should be noted that the SDE method has an advantage if the trans-

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port equation has to be solved in multi-dimensions. In practice, the importance of upstream inhomogeneity for understanding of cosmic-ray acceleration at supernova remnants has been pointed out by various authors (e.g., Inoue et al., 2012). In this case, it is clear to consider the particle acceleration in three dimensions.

The simple-minded application of the SDE method has problems in actual numerical integration. First, δ-functions appear in SDE if we apply it to the shock front, where the background fluid velocity as well as the diffusion coefficient have a sudden jump. In order to avoid this, the shock structure is artificially smoothed (Achterberg and Krülls, 1992). However, even in this case, the time step has to be small enough for the simulated particles not to miss the sharp gradient at the shock front, which significantly slows down the simulation. Furthermore, in actual simulation time, approximation of the smooth shock transition causes incorrect particle spectrum. This difficulty was solved by Zhang (2000) who used the skew Brownian motion (Harrison and Shepp, 1981) which can be solved by a scaling method that eliminated the δ-functions in the SDE. Other numerical schemes to resolve this problem have also been proposed (Marcowith and Kirk, 1999; Achterberg and Schure, 2011). Second problem is that a large dynamic range in particle momentum causes low statistical accuracy at large momenta. This difficulty was also resolved by employing a particle splitting technique (Yoshida and Yanagita, 1994).

In this paper, we first attempt to perform simulations of electron acceleration incorporating *both* methods of Zhang (2000) and particle splitting. Owing to newly developed code, simulated spectra have cutoff shape accurate enough to be compared with analytical formulae. As a first step, we focus on the cases of one-dimensional plane shock. Extended studies for more complicated cases such as time dependent free escape boundary, nonuniform magnetic fields, and/or multi-dimensional systems (including spherical shock geometry) are simple but remained as future works.

2. Basic equations and numerical method

2.1. Basic equations

In this paper, we consider one-dimensional system, that is, all quantities depend on the spatial coordinate x. The diffusion–convection equation with energy-loss process is given by

$$\frac{\partial f}{\partial t}+\frac{\partial}{\partial x}\left(vf-K\frac{\partial f}{\partial x}\right)+\frac{1}{p^2}\frac{\partial}{\partial p}\left[\left(-\frac{p}{3}\frac{dv}{dx}+\frac{dp}{dt}\right)p^2 f\right]=0, \tag{1}$$

where $f(x,p,t)$ is the distribution function for electrons, and p is the electron momentum. Functions $v(x)$ and $K(x,p)$ are background velocity field and the spatial diffusion coefficient of the electrons, respectively. In this paper, we consider the synchrotron cooling. Then, the loss term becomes

$$\frac{dp}{dt}=-\beta_{syn}\gamma p, \tag{2}$$

where

$$\beta_{syn}=\frac{\sigma_T B^2}{6\pi m_e c}, \tag{3}$$

and $\gamma=\sqrt{(p/m_e c)^2+1}$ is the electrons' Lorentz factor, and B is the magnetic field. Physical constants, σ_T, m_e and c are Thomson cross section, mass of electron and velocity of light, respectively.

Introducing new quantities,

$$u=\ln\left(\frac{p}{m_e c}\right), \tag{4}$$

and

$$F(x,u,t)=p^3 f(x,p,t), \tag{5}$$

Eq. (1) becomes the Fokker–Planck form,

$$\frac{\partial F}{\partial t}+\frac{\partial}{\partial x}\left[\left(v+\frac{\partial K}{\partial x}\right)F\right]-\frac{\partial^2}{\partial x^2}(KF)-\frac{\partial}{\partial u}\left[\left(\frac{1}{3}\frac{dv}{dx}+\beta_{syn}\gamma\right)F\right]=0. \tag{6}$$

This equation is equivalent to the following SDEs of the Itô form:

$$dx=\left(v+\frac{\partial K}{\partial x}\right)dt+\sqrt{2K}dW, \tag{7}$$

$$du=-\left(\frac{1}{3}\frac{dv}{dx}+\beta_{syn}\gamma\right)dt, \tag{8}$$

where dW is a Wiener process given by the Gaussian distribution:

$$P(dW)=\frac{1}{\sqrt{2\pi dt}}\exp(-dW^2/2dt). \tag{9}$$

Numerical simulation by SDEs is much faster than that with the usual Monte Carlo method and is much easier than solving the original Fokker–Planck equation, because the SDEs are ordinary differential equations.

2.2. Method of Zhang (2000)

The application of the SDEs, Eqs. (7) and (8), for the study of electron acceleration at the shock is not simple, because the velocity field $v(x)$ has a sudden jump at the shock front, so that dv/dx in Eq. (8) contains δ-function. Similarly, if the diffusion coefficient also behaves discontinuously at the shock front, then $\partial K/\partial x$ in Eq. (7) also contains the δ-function. We take the comoving frame with the shock which is located at $x=0$ and we define $x<0$ as upstream region. Following Zhang (2000), we decompose the velocity field v and the diffusion coefficient K into two parts:

$$v(x)=v_c(x)+\frac{\Delta V}{2}\mathrm{sign}(x), \tag{10}$$

$$K(x)=K_c(x)+\frac{\Delta K}{2}\mathrm{sign}(x), \tag{11}$$

where $\Delta V=v(0^+)-v(0^-)$ and $\Delta K=K(0^+)-K(0^-)$, and sign(x) is the sign of x. Functions $v_c(x)$ and $K_c(x)$ are continuous for arbitrary x (including $x=0$). We scale the x coordinate according to its sign in the following way (Harrison and Shepp, 1981):

$$y=xs(x)=x\times\begin{cases}\alpha & (x<0)\\ \frac{1}{2} & (x=0)\\ 1-\alpha & (x>0),\end{cases} \tag{12}$$

where

$$\alpha=\frac{K(0^+)}{K(0^+)+K(0^-)}. \tag{13}$$

Then, SDEs (7) and (8) can be rewritten as

$$dy=s(x)\left[\left(v(x)+\frac{\partial K_c}{\partial x}\right)dt+\sqrt{2K}dW\right], \tag{14}$$

$$du=-\left(\frac{1}{3}\frac{dv_c}{dx}+\beta_{syn}\gamma\right)dt-\frac{\Delta V}{3\Delta K}[dx-s^{-1}(y)dy]. \tag{15}$$

Derivation of Eqs. (14) and (15) is the same way as of Zhang (2000). These equations do not contain δ-functions and can be integrated directly. Once $y(t)$ is obtained, the position of electrons $x(t)$ can be obtained by