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Variable inertia method: A novel numerical method for mantle convection simulation

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HIGHLIGHTS

- A novel method which enable us to solve the mantle convection explicitly is proposed.
- The results of a novel method are in good agreement with standard benchmark test.
- A novel method is suitable for parallel computations since it does not require global communication.

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ABSTRACT

3D numerical simulations have been very useful for the understanding of mantle convection of the earth. In almost all previous simulations of mantle convection, the (extended) Boussinesq approximation has been used. This method is implicit in the sense that buoyancy force and viscosity are balanced, and allows the use of long timesteps that are not limited by the CFL condition. However, the resulting matrix is ill-conditioned, in particular since the viscosity strongly depends on the temperature. It is not well-suited to modern large-scale parallel machines.

In this paper, we propose an explicit method which can be used to solve the mantle convection problem. If we can reduce the sound speed without changing the characteristics of the flow, we can increase the timestep and thus can use the explicit method. In order to reduce the sound speed, we multiplied the inertia term of the equation of motion by a large and viscosity-dependent coefficient. Theoretically, we can expect that this modification would not change the flow as long as the Reynolds number and the Mach number are sufficiently smaller than unity. We call this method the variable inertia method (VIM).

We have performed an extensive set of numerical tests of the proposed method for thermal convection, and concluded that it works well. In particular, it can handle differences in viscosity of more than five orders of magnitude.

1. Introduction

It is important to understand mantle convection in the earth because it drives plate tectonics and provides information of the thermal history of the earth. Mantle convection has two unique characteristics which make numerical simulations very difficult. First, the viscosity of the mantle is very high. As a result, the timescale of the convection is many orders of magnitude longer than the dynamical timescale. Second, the viscosity is not only high, but also strongly dependent on the temperature. It varies by more than ten orders of magnitude.

In almost all previous simulations of mantle convection, two approximations have been used. The first one is to ignore the inertia term. In this approximation, we ignore the inertia term in the equation of motion because the Prandtl number (*Pr*) is very large. In other words, we solve the balance between buoyancy force and viscosity. The second one is the (extended) Boussinesq approximation. In the case of the extended Boussinesq approximation, the thermal equation includes the effect of adiabatic compression, but the equation of motion or continuity equation doesn't. The approximated equation of motion is solved by using an iterative solver with an implicit manner. The timestep is generally evaluated by using the energy equation and it is much longer than that

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evaluated using the CFL and von-Neumann conditions (Zhong et al., 2007). Therefore, we can use a large timestep.

In order to solve mantle convection, several methods have been proposed. In early studies of mantle convection, the finite difference (FD) method has been widely used. For example, Torrance and Turcotte (1971) and McKenzie et al. (1974) used the second-order FD method, and (Christensen, 1984) solved variable viscosity fluid using an FD method. The finite element (FE) method is effective for solving problems with complicated geometry. King et al. (1990) developed ConMan code and (Moresi and Gurnis, 1996) developed Citcom code using the FE method. These codes are widely used in the field of mantle convection study. The SIMPLER algorithm is an efficient scheme based on the finite volume (FV) method, and (Ogawa et al., 1991) used this algorithm and developed a 3D mantle convection code. Using the FV method, (Kameyama et al. 2005) developed the ACuTE code, in which pseudo-compressibility and local time stepping techniques are used. Methods for 3D convection use iterative solvers. Since the iterative method requires global communication, it is difficult to achieve high efficiency on large-scale parallel supercomputers. Furthermore, high resolution calculation is very hard since convergence is slower for higher resolutions. The grid size of one of the highest resolution calculations is 512 by 512 by 128 (Kameyama and Yuen, 2006).

This problem of the implicit method exists not only in the simulation of mantle convection, but also in any simulation of subsonic flows. In the case of flows with high Reynolds numbers, the so-called reduced speed of sound technique (RSST) is gaining popularity (Rempel, 2005). The basic idea of RSST is simply to make the fluid more compressible, instead of applying the approximation of incompressibility. We know that the incompressible approximation can be applied to the original set of equations. As long as the flow is subsonic, we can apply the same incompressible approximation to the set of equations modified by RSST. Therefore, the set of equations modified by RSST should give a solution close to that of the original set of equations.

Hotta et al. (2012) applied RSST to the thermal convection zone of our Sun, and successfully performed the simulation with a resolution of 512 by 1024 by 3072 (Hotta et al., 2014). The previous record size simulation of the convection zone of our Sun, with the standard incompressible assumption, was 257 by 1024 by 2048 (Miesch et al., 2008).

In the case of mantle convection, it is not enough to just apply RSST, since the Mach number (M) and the Reynolds number (Re) are both very small. Only the Mach number is increased by RSST. Therefore, we need a different approach.

In this paper we present our new approach, the variable inertia method (VIM), with which we can increase the Mach number and Reynolds number simultaneously, without changing the characteristics of the convection by keeping the Rayleigh number (Ra) unchanged. As its name suggests, the basic idea of our method is to increase the inertia term (the left hand side) of the equation of motion. In the previous simulations of mantle convection, this term has been neglected, resulting in the instantaneous balance between the buoyancy term and the viscosity. The fact that we can ignore the inertia term suggests that we can *increase* it, without changing the behavior of the solution.

Increasing the inertia term does increase both the Mach number and the Reynolds number, but there is no guarantee that we can simultaneously bring both of these two numbers close to unity. Therefore, we need another parameter, which changes the Mach number and Reynolds number in a different way. We can achieve this by scaling the viscosity coefficient and thermal conductivity coefficient in a consistent manner, so that the Rayleigh number is unchanged. Thus, we can make both the Mach number and Reynolds number quite close to unity, and thereby make it possible to apply an explicit method to the mantle convection problem. In doing so, we also change the Prandtl number by many orders of magnitude. We found that it seems necessary to keep $Pr \gg 1$.

In this paper, we describe the variable inertia method for a basic set of equations of mantle convection simulations. In Section 2, we present equations for mantle convection. In Section 3, we describe our new method, the variable inertia method, in detail. In Section 4, we describe the smoothed particle hydrodynamics (SPH) formulation we used to discretize the modified equations. In Section 5, we show the results of numerical tests. In Section 6, we present the results of the standard benchmark test proposed by Blankenbach et al. (1989), and show the results of VIM with FD and SPH. In Section 7, we summarize our results.

2. Basic equations

2.1. Governing equations for mantle convection

The following set of equations describes the mantle convection,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0, \tag{1}$$

$$\rho \frac{d\boldsymbol{v}}{dt} = -\nabla p + \rho \boldsymbol{g} + \nabla \cdot \boldsymbol{\Pi}, \tag{2}$$

$$\rho c_p \frac{dT}{dt} - \alpha T \frac{dp}{dt} = \nabla \cdot (k \nabla T) + (\boldsymbol{\Pi} \cdot \nabla) \cdot \boldsymbol{v}, \tag{3}$$

where ρ is the density, t is the time, $\boldsymbol{v}$ is the velocity, p is the pressure, $\boldsymbol{g}$ is the gravity, c_p is the specific heat at constant pressure, T is the temperature, α is the coefficient of thermal expansion, k is the coefficient of thermal conductivity, and $\boldsymbol{\Pi}$ is the viscous stress tensor. The definition of Π is given by

$$\Pi_{ij} = \mu \left[\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3} (\nabla \cdot \boldsymbol{v}) \delta_{ij} \right], \tag{4}$$

where μ is the coefficient of viscosity and i and j indicate the directions in the Cartesian coordinates. In this paper, we mainly use the Lagrangian form, since we use a particle method to discretize the basic equation. We can also apply the same transformation to the Eulerian form as will be shown in Section 6.4.

For the equation of state, we used the Tillotson equation of state (Tillotson, 1962), which is expressed by

$$p = \left(a_0 + \frac{b_0}{\dfrac{u}{u_0 \eta_0^2} + 1} \right) \rho u + A_0 \eta_1 + B_0 \eta_1^2. \tag{5}$$

Here u is the internal energy and a_0, b_0, u_0, A_0, and B_0 are material parameters. For the test problems, we used parameters of granite, where $a_0 = 0.5$, $b_0 = 1.3$, $u_0 = 16$ MJ/kg, $A_0 = 18$ GPa, and $B_0 = 18$ GPa. Here, η_0 and η_1 are defined as

$$\eta_0 = \frac{\rho}{\rho_0}, \tag{6}$$

$$\eta_1 = \eta_0 - 1, \tag{7}$$

where $\rho_0 = 2680$ kg/m^3 for granite. We assumed that the internal energy of the mantle is proportional to the temperature,

$$du = c_p dT. \tag{8}$$