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Fractal structures for the Jacobi Hamiltonian of restricted three-body problem

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H I G H L I G H T S

- Poincaré section of survival orbits in binary stars has a strange repeller.
- Density of survival particles has a spiral form.
- Fractal dimension of the strange repeller is similar to galaxy fractal dimension.
- Survival probability drops algebraically due to stability islands.

A R T I C L E I N F O

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A B S T R A C T

We study the dynamical chaos and integrable motion in the planar circular restricted three-body problem and determine the fractal dimension of the spiral strange repeller set of non-escaping orbits at different values of mass ratio of binary bodies and of Jacobi integral of motion. We find that the spiral fractal structure of the Poincaré section leads to a spiral density distribution of particles remaining in the system. We also show that the initial exponential drop of survival probability with time is followed by the algebraic decay related to the universal algebraic statistics of Poincaré recurrences in generic symplectic maps.

1. Introduction

The restricted three-body problem was at the center of studies of dynamics in astronomy starting from the works of Euler (1772), Jacobi (1836) and Poincaré (1890). The progress in the understanding of this complex problem in XXth and XXIth centuries is described in the fundamental books (Szebehely, 1967; Hénon, 1997; 2001; Valtonen and Karttunen, 2006). As it was proven by Poincaré (1890) in the general case this system is not integrable and only the Jacobi integral is preserved by the dynamics (Jacobi, 1836). Thus a general type of orbits has a chaotic dynamics with a divided phase space where islands of stability are embedded in a chaotic sea (Chirikov, 1979; Lichtenberg and Lieberman, 1992; Ott, 1993).

In this work we consider the Planar Circular Restricted Three-Body Problem (PCRTBP). This is an example of a conservative Hamiltonian system (in a synodic or rotating reference frame of two binaries) with two degrees of freedom. However, this is an open system since some trajectories can escape to infinity (be ionized) so that the general theories of leaking systems (Altmann et al., 2013) and naturally open systems (e.g. Contopoulos and Efstathiou, 2004) find here their direct applications. It is known that such open systems are characterized by strange repellers related to non-escaping orbits and by an exponential time decay of probability to stay inside the system. However, as we show, in the PCRTBP system with a divided phase space one generally finds an algebraic decay of probability of stay related to an algebraic statistics of Poincaré recurrences in Hamiltonian systems (see e.g. Chirikov and Shepelyansky, 1981; Karney, 1983; Chirikov and Shepelyansky, 1984; Meiss and Ott, 1985; Chirikov and Shepelyansky, 1999; Cristadoro and Ketzmerick, 2008; Shevchenko, 2010; Frahm, and Shepelyansky, 2010, and Refs. therein). This effect appears due to long sticking of trajectories in a vicinity of stability islands and critical Kolmogorov–Arnold–Moser (KAM) curves. Thus an interplay of fractal structures and algebraic decay in the PCRTBP deserves detailed studies.

Among the recent studies of the PCRTBP we point out the advanced results of Nagler (2004, 2005) where the crash probability

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dependence on the size of large bodies has been studied and the fractal structure of non-escaping orbits has been seen even if the fractal dimensions were not determined. This research line was extended in Astakhov and Farrelly (2004); Astakhov et al. (2005) with a discussion of possible applications to the Kuiper-belt and analysis of various types of orbits in Barrio et al. (2009); Zotos (2015). The analysis of orbits in three dimensional case is reported in Makó et al. (2010) and basin of escaping orbits around the Moon has been determined in de Assis and Terra (2014).

In this work we determine the fractal dimension of non-escaping orbits for the PCRTBP with comparable masses of heavy bodies and consider the properties of Poincaré recurrences and the decay probability of stay in this system. The system description is given in Section 2, the structure of strange repeller is analyzed in Section 3, the decay of Poincaré recurrences and probability of stay are studied in Section 4, a symplectic map description of the dynamics is given in Section 5, discussion of the results is presented in Section 6.

2. System description

The PCRTBP system is composed of a test particle evolving in the plane of a circular binary whose primaries have masses $m_1 = 1 - \mu$ and $m_2 = \mu$ with $m_1 > m_2$. In the synodic frame the dynamics of the test particle is given by the Hamiltonian

$$H(p_x, p_y, x, y) = \frac{1}{2}\left(p_x^2 + p_y^2\right) + y p_x - x p_y + V(x, y) \tag{1}$$

where x and y are the test particle coordinates, $p_x = \dot{x} - y$ and $p_y = \dot{y} + x$ are the corresponding canonically conjugated momenta, and

$$V(x, y) = -\frac{(1-\mu)}{\left((x-\mu)^2 + y^2\right)^{1/2}} - \frac{\mu}{\left((x + (1-\mu))^2 + y^2\right)^{1/2}} \tag{2}$$

is the gravitational potential of the two primaries. Here the distance between primaries is 1, the total mass $m_1 + m_2 = 1$, the gravitational constant $\mathcal{G} = 1$, consequently the rotation period of the binary is 2π. Hamiltonian (1) with potential (2) represents the Jacobi integral of motion (Jacobi, 1836). In the following we define the Jacobi constant as $C = -2H$. This Jacobi Hamiltonian describes also the planar dynamics of an electrically charged test particle experiencing a perpendicular magnetic field and a classical hydrogen-like atom with a Coulomb-like potential (2).

We aim to study the dynamics of particles evolving on escaping and non-escaping orbits around the binary. We perform intensive numerical integration of the equations of motion derived from Hamiltonian (1) using an adaptive time step 4th order Runge–Kutta algorithm with Levi-Civita regularization in the vicinity of the primaries (Levi-Civita, 1920). The achieved accuracy is such as the integral of motion relative error is less than 10^{-9} (10^{-5}) for more than 91% (99%) of integration steps. For different Jacobi constants C, we randomly inject up to 10^8 test particles in the $1.3 \leq r \leq 2.5$ ring with initial radial velocity $\dot{r} = 0$ and initial angular velocity $\dot{\phi} < 0$ (r and ϕ are polar coordinates in the synodic frame). Each test particle trajectory is followed until the integration time attains $t_S = 10^4$ or until the region $r > R_S = 10$ is reached where we consider that test particles are escaped (ionized) from the binary.

3. Strange repeller structures

In phase space, orbits are embedded in a three-dimensional surface defined by the Jacobi constant C. In order to monitor particle trajectories we choose a two-dimensional surface defined by an additional condition. Here we choose either the condition ($\dot{r} = 0, \dot{\phi} < 0$) to represent Poincaré section as a (x, y)-plane (Figs. 1, 5–7 and 10) or the condition ($y = 0, p_y > 0$) to represent Poincaré section as a (p_x, x)-plane (Fig. 2). A similar approach was also used in Nagler (2004, 2005).

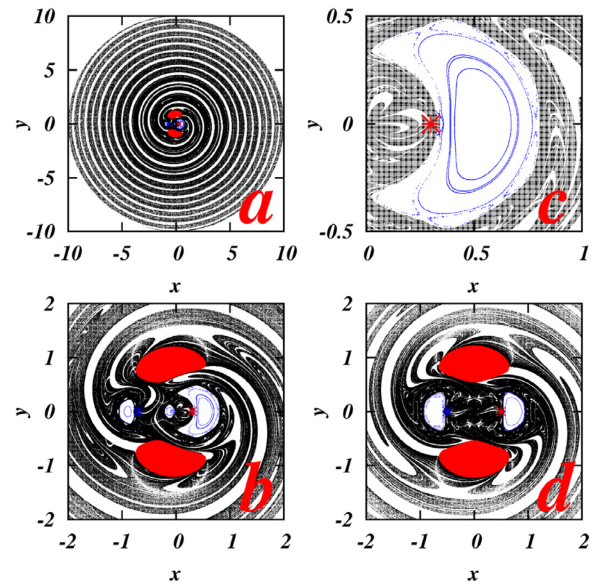

Fig. 1. (x, y) – Poincaré sections of the Jacobi Hamiltonian (1) with $\dot{r} = 0$ and $\dot{\phi} < 0$. Poincaré sections for a binary with $\mu = 0.3$, $C = 3$ are shown in panels: (a) at a large scale, (b) at an intermediate scale, (c) close-up in the vicinity of the primary mass. Panel (d) shows the Poincaré section for $\mu = 0.5$ and $C = 3$. Red regions are forbidden since there $\dot{x}^2 + \dot{y}^2 < 0$. Black dots represent non-escaped orbits staying inside the $r < R_s = 10$ region after time $t = 10$. Invariant KAM curves (blue dots) are obtained choosing initial conditions inside KAM islands. The red (blue) star $*$ ($*$) gives the position of the $1 - \mu$ mass (μ mass). The Poincaré section is obtained with orbits of $N = 10^7$ test particles initially placed at random in the region $1.3 \leq r \leq 2.5$. Particles as escaped once $r > R_S$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article).

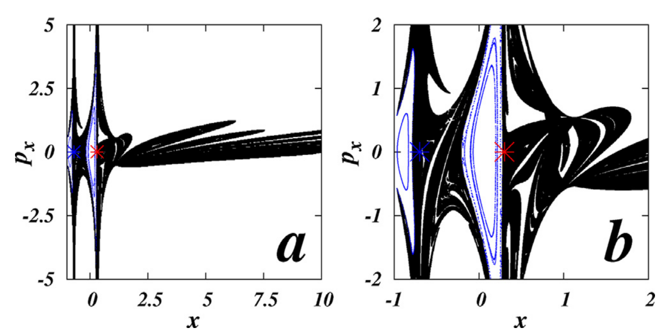

Fig. 2. (p_x, x) – Poincaré section of the Jacobi Hamiltonian (1) with $y = 0$ and $p_y > 0$ for a binary with $\mu = 0.3$, $C = 3$ (corresponding to Fig. 1a, b, c). Panel (a): Poincaré section at large scale; panel (b): zoom in the vicinity of primaries. Black dots represent non-escaped orbits staying inside the $r < R_s = 10$ region after time $t = 10$. Blue dots represent bounded orbits inside stability islands. The red (blue) star $*$ ($*$) gives the position of the primary (secondary) mass as in Fig. 1. The Poincaré section is obtained with the same orbits as in Fig. 1. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article).

We show in Fig. 1 (panels a, b, c) an example of (x, y) – Poincaré section of the Jacobi Hamiltonian (1) for mass parameter $\mu = 0.3$ and Jacobi constant $C = 3$. Red regions correspond to forbidden zones where particles would have imaginary velocities. Inside central islands in the close vicinity of primaries blue points mark out regular and chaotic orbits of bounded motion. In particular, the KAM invariant curves (Lichtenberg and Lieberman, 1992) can be seen *e.g.* in Fig. 1c. In Fig. 1a, the trace of non-escaped chaotic orbits (black points) remaining inside the disk $r < R_S = 10$ after time $t = 10$ defines a set of points forming two spiral arms centered on the binary center of mass. This set has a spiral structure of strange repeller since orbits in its close vicinity rapidly move away from