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Analytical representations for simple and composite polytropes and their moments of inertia

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HIGHLIGHTS

- Simple, approximate analytical forms are provided for polytropes.
- Moment of inertia is shown to be a simple function of polytropic index.
- A composite polytrope is constructed for application to exoplanets.

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ABSTRACT

Polytropes are widely applied in astrophysics. To facilitate their use, we derive analytical formulae for the moment of inertia as a function of polytropic index. We also provide 1- and 3-parameter equations that replicate the density variations in polytropic bodies to varying degrees of accuracy, determined by numerical calculations and analytical results for polytropic indices between 0 and 5. As an example, we construct a composite polytrope, suitable for gas giants, exoplanets, or tiny sub-solar dwarfs, wherein an inner sphere is modeled by constant density, which represents the density jump associated with production of a relatively incompressible solid, and an outer envelope is modeled as having a polytropic index near 2.5, which corresponds to a diatomic gas. Envelope sizes are constrained by the moment of inertia.

1. Introduction

Polytropes are the underpinning of our theoretical understanding of stellar structure and evolution (after Chandrasekhar, 1939) and find wide use in other areas of astrophysics (summarized by Horedt, 2004). For example, polytropic equations have been used to describe mass and position of planets and moons in solar and satellite systems, respectively (e.g. Geroyannis and Dallas, 1994), and to investigate globular clusters (e.g. Nguyen and Pedraza, 2013), collapsing molecular clouds and Bok globules (Curry and McKee, 2000), quark stars (Lai and Xu, 2009), and the effect of pressure anisotropy on compact objects (Herrara and Baretto, 2013). Furthermore, polytropes have been applied to problems of stability and oscillation (e.g., Gleiser and Sowinski, 2013; Breysse et al., 2014) as well as to address relativistic effects (e.g., Geroyannis and Karageorgopoulos, 2014). However, use of polytropes can be cumbersome because representations are typically numerical, whereas analytical expressions are desired for many applications (e.g., Nguyen and Lingam, 2013).

The polytropic construct is based on the hydrostatic equation, which relates pressure to gravitation, and on Poisson's equation which describes gravitational attraction. In the absence of rotation, which we neglect here, symmetry is spherical. To specify all three variables (ϕ = gravitational potential; ρ = density; and P = pressure) as a function of radius (r), a relationship between pressure and density is assumed:

$$P = \kappa\rho^{\gamma} \quad (1)$$

where κ and γ are constants (e.g., Emden, 1907; Eddington, 1959). Eq. (1) has the same form as the equation for a perfect gas under adiabatic conditions:

$$\left(\frac{P}{P_0}\right) = \left(\frac{\rho}{\rho_0}\right)^{C_P/C_V} = \left(\frac{T}{T_0}\right)^{C_P/R} \quad (2)$$

where C_P and C_V are the heat capacities under constant pressure (isobaric) and constant volume, V, (isochoric) conditions, respectively. Also, T is temperature and R is the gas constant. Hence,

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γ is referred to as the adiabatic index. Knowing density constrains pressure via Eq. (2) and for potential:

$$\varphi = \frac{(\gamma)}{\gamma - 1}\frac{P}{\rho}. \quad (3)$$

Changes of variables make the problem tractable: using the polytropic index $n = 1/(\gamma - 1)$ and non-dimensionalizing the variables provides the Lane–Emden equations which can be solved analytically for $n = 0$, 1, or 5. Numerical solutions are tabulated for many other cases (e.g., Emden, 1907). The case of $n = 0$ corresponds to constant density and $n = 5$ to a density distribution that extends to infinity, although the mass is finite. These two cases set bounds for depicting stars and planets. Different indices are applied to the various star types: e.g., white dwarfs are modeled by high n, whereas those with convective cores are modeled with low n, and neutron stars with very low n (e.g., Ferrari et al., 2010). An important case is $n = 2.5$ which corresponds to a diatomic gas (e.g. H_2; e.g., Maron and Prutton, 1970) under adiabatic conditions. Similarly, under adiabatic conditions a monatomic gas (e.g. He) or a triatomic gas (e.g. CO_2) respectively correspond to the polytropes with $n = 1.5$ and $n = 3.5$.

Early work on the gas giant planets found that a polytropic index of $n \sim 1$ fit various properties (e.g., Bobrov et al., 1978). However, the outermost region of Jupiter is largely hydrogen gas and thus should instead be described by $n \cong 2.5$. In recognition of such disparities, more complicated approaches have been pursued to model planets (e.g., Horedt and Hubbard, 1983) and, more recently, exoplanets. For example, modifications of the polytrope of the form $\rho = \rho_0 + aP^b$ (where a and b are fitting parameters) have been applied to solid exoplanets with masses up to 40 times that of Earth (Seager et al. 2007). However, it is presently difficult to constrain polytropes of exoplanets because mass and radius are their only known physical parameters (e.g., Baraffe et al., 2010; Swift et al., 2012). Recently, advances have been made in determining the parameter J_2 from transiting exoplanets (Carter and Winn, 2010; Leconte et al., 2011). The parameter J_2 is related to the moment of inertia (I), which reflects the interior compression and structure because I is strongly affected by concentration of mass near the center.

In anticipation of further advances in measuring the physical properties of exoplanets, and to provide analytical forms for polytropes that can be used in other astrophysical applications, the present paper explores two-layer (composite) polytropic models and the connection with the moment of inertia. The example considered is exoplanets. This exercise is appropriate for several reasons. (1) Compositions of exoplanets are unknown, yet determine the equation-of-state. (2) Uncertainties exist even in models of interiors of the gas giants, for which considerably more information is available (see e.g., Gaulme et al., 2011), This situation persists because the data on metallic hydrogen, which is considered to be an important interior phase, is limited to one high pressure–high temperature point from transient, shock experiments (Nellis, 2013). Although predictions starting with Wigner and Huntington (1935) and continuing to the present suggest stability of metallic hydrogen at ambient temperature and very high pressure, diamond anvil experiments (e.g. Eremets and Trojan, 2011) have not provided convincing evidence of this transition (Nellis, 2013). Equation-of-state models are therefore not fully consistent with experiments. (3) For the hot Jupiter type of exoplanet, a significant part of the outer layers should be gas, which is amenable to polytropic equations of state.

We begin with a general formula for moment of inertia as a function of n and with simple and useful representations for the density of polytropes. Earlier approximations for polytropes involved polynomials but accuracy in such approaches requires a large number of terms (>25: e.g., Hunter, 2001; Saad, 2004). Reasonable approximations of various polytropes with indices from 0 to 500 have been made with 4 parameters (Liu, 1996). We show here that equations for density can be represented reasonably well by 1 or 3 parameters, which will simplify use of polytropes (see Horedt, 2004 for a summary of the diverse applications).

We construct a composite polytrope where the inner spherical region is assumed to have a constant density that is higher than ρ of the outer spherical shell. This approach represents gassy exoplanets, which have a large solid interior that is incompressible in comparison to the gaseous, outer envelope, and significantly denser, due to this phase transition being pressure-induced. The presence of additional interior structure such as an innermost rock + metal core (see figures in Lodders and Fegley, 1998) is neglected here, as its effect will be small, as gauged by Jupiter whose core is only ~3% of the planet's mass (e.g., Saumon and Guillot, 2004). Composite polytropes have been applied to stars (e.g., Bejger, 2005) but these differ considerably from our construct because continuously changing density was assumed, whereas in a planet or perhaps in the coldest, smallest stars, the density should be discontinuous at the transition between molecular and metallic species. Other composite models for stars (e.g., Ruciński, 1988, which is based on the model of Rappaport et al., 1983; see also the summary by Curry and McKee, 2000) assume in addition to continuity in density that the inner region is more compressible than the outer, which is not the case for a planet wherein transitions are pressure driven: instead n of the gassy envelope should be lower than n of the solid sphere). Various equations-of-state approaches have been used to represent planetary interiors (see e.g., Horedt, 2004). We show that moment of inertia of Jupiter can be reproduced by the combination of an incompressible central sphere (i.e. $n = 0$) with an outer molecular mantle with polytropic index near 2.5, which approximates behavior of a perfect diatomic gas, and suggests a simple, generic polytrope for investigation of gassy exoplanets.

2. Calculations

2.1. Moments of inertia for polytropes

Tables of density as a function of reduced radius for different polytropes were downloaded from the web using the polytrope calculator hosted by Clemson University (Brown et al., 2006). Moments of inertia (I) are calculated on the basis of spherical shells using:

$$I = \frac{2}{3}\int_0^a 4\pi r^4 \rho(r)dr \quad (4)$$

where a is the surface radius. Simple solutions exist where density (ρ) can be obtained analytically: The reduced (normalized) moment of inertia [$I_{norm} = I/(Ma^2)$], is well known for $n = 0$, equaling 2/5. We obtain $I_{norm} = 2(\Pi^2 - 6)/(3\pi 2)$ for $n = 1$, and $I_{norm} = 0$ for $n = 5$ by taking the limit of a approaching infinity. These three simple solutions (Fig. 1) are fit by

$$\frac{I}{Mr^2} = \left[\frac{5}{6}\left(\frac{\pi^2 - 6}{\pi^2}\right) - \frac{8}{25}\right](5 - n) + \left[\frac{2}{25} - \left(\frac{\pi^2 - 6}{6\pi^2}\right)\right](5 - n)^2 \quad (5)$$

For comparison, we numerically integrated Eq. (4) for these three cases and many other values of n between 0.5 and 4.99. As shown in Fig. 1, the numerical results differ only slightly from the analytical results for $n = 0$, 1 and 5, presumably due to finite spacing in the tabular output and the approximations used in the numerical recipe. Similarly, the numerical results for other values of n differ only slightly from Eq. (5).