Simple algorithms for relative motion of satellites

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HIGHLIGHTS

- We derived a numerical method for the computation of two-satellite relative motion.
- We derived a set of analytical solutions for the two-body variational equations.
- The numerical method and the analytical solution coincide.

ARTICLE INFO

ABSTRACT

We present an accurate simple-to-implement numerical method for the computation of the relative motion of two-satellites. This is presented for both the linearized approximation and in an exact formulation. We also derive a basic set of analytical solutions for the variational equations of the two-body motion. This is shown to be a useful approximation for the relative motion of two-satellites even in the so called J_2 problem, provided one uses the secular J_2-theory to obtain the orbit precession. The numerical method results are compared with approximations produced by the two-body variational but precessed approximation. We find a good agreement for quasi-circular orbits with same inclination.

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1. Introduction

We consider formation flight of satellites in proximity, in fact so close to each other that the linearized approximation gives reasonable results. Clohessy and Wiltshire (1960) solved this problem but only in the case of circular Keplerian orbit. A possibility to obtain the first order relative motion is to differentiate the formulae for the Kepler motion. One example of the formulation of such an algorithm can be found in Mikkola and Innanen (1999) and also Mikkola et al. (2000), Mikkola et al. (2002). There are many publications discussing formation flight (Alfriend et al., 2001; Roscoe et al., 2013; Schaub and Alfriend, 2001), which handle the problem in terms of orbital elements and their perturbations. More recently Kristiansen et al. (2010) published a new formulation which is valid for eccentric orbits also. We try to present the results in a more elementary and easy to understand way and consider in detail quasi-circular orbits. One new result in this paper is a numerical method that is simple to implement and gives accurate results for the orbits.

We consider two methods: In Section 3 we examine the two-body orbit with precessions obtained by solving the secular Hamiltonian in which the J_2 term is included. The results show that this method is useful to determine the relative trajectories of two satellites at least for the mission planning phase, but the method may suffer from phase errors especially if osculating elements are used instead of the mean elements (Walter, 1967; Gupta et al., 2011). For accurate computations we suggest (in Section 4) a method that uses logarithmic Hamiltonian leapfrog. This has the advantage of simplicity of programming and possibility of high precision.

2. A simple set of independent solutions for the variational problem of two-body motion

Let us consider the two-body problem in units in which $G = 1$, i.e. for the sake of simplicity we write the equations of motion in the form

$$\dot{\mathbf{r}} = \mathbf{v}, \quad \dot{\mathbf{v}} = -m\mathbf{r}/r^3, \tag{1}$$

where $\mathbf{v}$ is the velocity, $\mathbf{r}$ is the position vector and $r = |\mathbf{r}|$ is the distance. The variational equations take the form

$$\dot{\mathbf{x}} = \mathbf{w}, \quad \dot{\mathbf{w}} = -m(\mathbf{x}/r^3 - 3\mathbf{r} \cdot \mathbf{xr}/r^5), \tag{2}$$

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where $\mathbf{x}$ is the variation of the position and $\mathbf{w} = \dot{\mathbf{x}}$ is the variation of velocity. Since the solutions of these equations can be used to approximate the relative motion of two satellites in almost the same orbit (if they stay close to each other), it is worth to study the various possibilities to formulate the solution of that equation.

It is obvious, and well known, that a partial derivative of two-body motion with respect to any orbital element is a particular solution of the variational equation (2). Let $q_k \; k =, 1, 2, \dots, 6$ be a set of orbital elements i.e.

$$\mathbf{q} = (M_0, i, \Omega, \omega, a, e),$$

which are the mean anomaly M at epoch, inclination, ascending node, argument of pericenter, semi-major-axis and eccentricity, respectively. The independent solutions for the variational equation (2) are thus

$$\mathbf{x}_k = \frac{\partial \mathbf{r}}{\partial q_k} \quad \text{and} \quad \mathbf{w}_k = \frac{\partial \mathbf{v}}{\partial q_k}. \tag{3}$$

This makes it possible to obtain relatively easily all the independent solution. However, the results would look quite complicated. We proceed as follows: infinitesimal shift along the orbit gives the velocity as a solution, any infinitesimal rotation gives the cross product of a constant vector and the position vector, change of the semi-major axis gives change of the length of the position vector as well as change of mean motion resulting to combination of position vector and time times velocity. The cross product of a constant vector and the position vector can be written as a linear combination of the cross products of the unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ and the position $\mathbf{r}$. Thus we have the individual solutions $\mathbf{i} \times \mathbf{r}$, $\mathbf{j} \times \mathbf{r}$ and $\mathbf{k} \times \mathbf{r}$. Finally the most complicated part is due to the effect of changing the eccentricity, but even this can be reduced to a rather simple form. Here we give the list of the results (note that the $\mathbf{x}_k$ and $\mathbf{w}_k$ here are not exactly any partial derivatives but constant factors have been excluded):

$$\begin{cases} \mathbf{x}_1 &= \mathbf{v} \\ \mathbf{x}_2 &= \mathbf{i} \times \mathbf{r} \\ \mathbf{x}_3 &= \mathbf{j} \times \mathbf{r} \\ \mathbf{x}_4 &= \mathbf{k} \times \mathbf{r} \\ \mathbf{x}_5 &= 2\mathbf{r} - 3t\mathbf{v} \\ \mathbf{x}_6 &= m(2 - (p+r)/a)\mathbf{r} - \mathbf{r} \cdot \mathbf{v}(p+r)\mathbf{v}, \end{cases} \tag{4}$$

where $p = (\mathbf{r} \times \mathbf{v})^2/m$ is the semi-latus-rectum, a is the semi-major axis $(1/a = 2/r - v^2/m)$ and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the standard unit vectors. For the derivatives $\mathbf{w}_k = \dot{\mathbf{x}}_k$ one gets

$$\begin{cases} \mathbf{w}_1 &= -m\mathbf{r}/r^3 \\ \mathbf{w}_2 &= \mathbf{i} \times \mathbf{v} \\ \mathbf{w}_3 &= \mathbf{j} \times \mathbf{v} \\ \mathbf{w}_4 &= \mathbf{k} \times \mathbf{v} \\ \mathbf{w}_5 &= -\mathbf{v} + 3t\mathbf{r}/r^3 \\ \mathbf{w}_6 &= (2v^2 - m/r - \dot{r}^2)\dot{r}\mathbf{r} + (m - rv^2)\mathbf{v}, \end{cases} \tag{5}$$

where $\dot{r} = \mathbf{r} \cdot \mathbf{v}/r$ is the radial derivative. The total expressions for $\mathbf{x}$ and $\mathbf{w}$ can now be written simply as

$$\mathbf{x} = \sum_{k=1}^{6} \alpha_k \mathbf{x}_k, \quad \mathbf{w} = \sum_{k=1}^{6} \alpha_k \mathbf{w}_k, \tag{6}$$

where the coefficients α_k are the constants of motion to be determined from initial conditions.

One notes that the expression are independent of orbit type, i.e. valid for ellipse, parabola and hyperbola without any reformulation.

The results obtained using only the two-body variational equations are valid only for a limited time and can thus be used

just to get a first idea of how to arrange a formation flight. For more accurate results it seems necessary to use numerical computations. In the next section we consider one new method for numerical integration of the difference of two nearby orbits.

3. Using secular J_2 motion

The secular Hamiltonian of the J_2 problem (Escobal, 1965; Stiefel et al., 1971)

$$H = -m/(2a) + J_2 \frac{m}{8} \frac{1 + 3\cos(2i)}{a^3(1 - e^2)^{3/2}}, \tag{7}$$

when written in terms of the Keplerian elements (and the square of Earth's radius in included in the value of J_2). The Lagrangian perturbation equations give the well known secular effects of the J_2 term as

$$\begin{aligned} \dot{\Omega} &= \frac{-3J_2 m \cos(i)}{2a^{7/2}(1 - e^2)^2} \\ \dot{\omega} &= 3\frac{J_2 m(-1 + 5\cos^2(i))}{4a^{7/2}(1 - e^2)^2} \\ \dot{M} &= \frac{3J_2 m(-1 + 3\cos^2(i))}{4a^{7/2}(1 - e^2)^{3/2}}, \quad (\dot{M} = n + \grave{M}), \end{aligned} \tag{8}$$

where $n = 1/(a\sqrt{a/m})$ is the mean motion and $\grave{M}$ means the J_2 perturbation effect in $\dot{M}$. In this approximation the elements a, e, i remain constants and thus the complete solution for the secular Hamiltonian can be written

$$\mathbf{r} = \widehat{\mathbf{G}} \mathbf{r}_{2B}(M), \tag{9}$$

where the mean anomaly M is obtained as $M = M(0) + t(n + \grave{M})$ and the coordinates of $\mathbf{r}_{2B}(M)$ are obtained in the normal way using Kepler's equation and the original unperturbed orbital elements. The matrix $\widehat{\mathbf{G}}$ gives effect due the Ω and ω precession, while the $\grave{M}$ effect changes the rate of the two-body motion $\mathbf{r}_{2B}(M)$. The precession matrix here consists of rotation around the z-axis by the angle due to Ω precession followed by a rotation in the orbital plane by the amount of ω-precession. Note that these rotations commute, i.e. their order does not matter. We use a form of the Euler's rotation formula

$$\mathbf{y} = \mathbf{f} + c_1\left(|\vec{\epsilon}|^2\right)\vec{\epsilon} \times \mathbf{f} + c_2\left(|\vec{\epsilon}|^2\right)\vec{\epsilon} \times (\vec{\epsilon} \times \mathbf{f}), \tag{10}$$

in which $\mathbf{f}$ is the rotated vector and $\mathbf{y}$ is the result. This operation rotates $\mathbf{f}$ around the vector $\vec{\epsilon}$ and the rotation angle is equal to the length of the vector $= |\vec{\epsilon}|$. These c_k's are the Stumpff-functions, which may be defined by Stumpff (1962), Stiefel et al. (1971)

$$\begin{aligned} \sin(\theta) &= \theta c_1(\theta^2) = \theta - \theta^3 c_3(\theta^2) = \theta - \theta^3/3! + \theta^5 c_5(\theta^2) \\ \cos(\theta) &= c_0(\theta^2) = 1 - \theta^2 c_2(\theta^2) = 1 - \theta^2/2! + \theta^4 c_4(\theta^2). \end{aligned} \tag{11}$$

Writing $z = \theta^2$ the c-functions can be expanded in power series as

$$c_n(z) = \sum_{j=0}^{\infty} \frac{(-z)^j}{(n + 2j)!}. \tag{12}$$

For small argument values this series can be used to evaluate the functions, while for large argument value the trigonometric expressions derivable from (11) are useful.

The effect of Ω precession can be obtained by taking $\vec{\epsilon}_\Omega = (0, 0, \Delta\Omega)$ and the ω-precession effect is produced be using the (present) values of $\mathbf{r}$ and $\mathbf{v}$ to evaluate $\vec{\epsilon}_\omega = \Delta\omega \mathbf{r} \times \mathbf{v}/|\mathbf{r} \times \mathbf{v}|$ and applying the operation (10) with both $\vec{\epsilon}_\Omega$ and $\vec{\epsilon}_\omega$. Since these are rotations the total result is a rotation as proved long ago by Euler (here the one in (9)).