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Computing rotating polytropic models in the post-Newtonian approximation: The problem revisited

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HIGHLIGHTS

- The "complex-plane strategy" is applied to the post-Newtonian approximation.
- The computations focus on the rotating polytropic models.
- Comparisons with previous results show improved accuracy.
- A "hybrid approximative scheme" is briefly discussed.
- This scheme seems to improve further the accuracy of the results.

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ABSTRACT

In this paper, the problem of computing uniformly rotating polytropic models in the post-Newtonian approximation is revisited by applying to its treatment the so-called "complex plane strategy". We achieve to remove certain difficulties, otherwise involved in the computations of general-relativistic polytropic models simulating rapidly rotating neutron stars, and to compute results of improved accuracy when compared to corresponding results of other reliable numerical methods.

1. Introduction

To study rapidly rotating neutron stars in hydrostatic equilibrium, we consider the relativistic and rotational effects as decoupled perturbations acting on a nonrotating Newtonian configuration obeying the polytropic "equation of state" (EOS, EOSs). The original contributions to such an approach, so-called "post-Newtonian approximation" (PNA), are due to Chandrasekhar (1965), Krefetz (1966, 1967), and Fahlman and Anand (1971).

In this study, we revisit the problem by applying to the computations the so-called "complex plane strategy" (CPS). This method consists in solving all differential equations involved in the PNA's computational scheme in the complex plane. Numerical integrations are carried out by the Fortran code dcrkf54.f95 (Geroyannis and Valvi, 2012), which is a Runge–Kutta-Fehlberg code of fourth and fifth order modified so that to integrate "initial value problems" (IVP, IVPs) established on systems of first-order "ordinary differential equations" (ODE, ODEs) of complex-valued functions in one complex variable along prescribed complex paths.

2. Basics of the post-Newtonian approximation

In the framework of PNA, the equation of hydrostatic equilibrium for a uniformly rotating relativistic configuration is written as (cf. (Krefetz, 1967), Eq. (1))

$$\left(1 - \frac{n+1}{c^2}\frac{p}{\rho}\right)\nabla p = \rho\,\nabla U_{\mathrm{eff}}. \tag{1}$$

where the effective potential U_{eff} is given by (cf. (Krefetz, 1967), Eq. (2))

$$U_{\mathrm{eff}} = U + \frac{1}{2}\Omega^2 l^2 + U_{\mathrm{rel}}, \tag{2}$$

and the relativistic potential U_{rel} has the form (cf. (Krefetz, 1967), right-hand side of Eq. (2))

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$$U_{\rm rel} = \frac{1}{c^2}\left[\left(\frac{\Omega^2 l^2}{2}\right)^2 + 4U\left(\frac{\Omega^2 l^2}{2}\right) + 2\Phi - 4\Omega^2 l^2 D\right]; \tag{3}$$

the potentials U, Φ, and D obey Eqs. (3a), (3b), and (4) of (Krefetz, 1967), respectively, ρ is the rest-mass density, p the pressure, Ω the uniform angular velocity, l the distance from the rotation axis, and c the speed of light.

In this study, we assume that the pressure p and the rest-mass density ρ obey the polytropic EOS

$$p = K\rho^{1+(1/n)}, \tag{4}$$

where K is the polytropic constant and n the polytropic index lying in the interval $n \in [0, 5) = \mathbb{I}_n \subset \mathbb{R}$ (the value $n = 5$ yields a model with infinite radius). The polytropic density function $\Theta(\xi, \mu)$, where $\mu = \cos(\vartheta)$, and the dimensionless length ξ are defined by Fahlman and Anand (1971) (Eq. (10))

$$\rho = \rho_c \Theta^n,$$
$$r = \left[\frac{(n+1)p_c}{4\pi G\rho_c^2}\right]^{1/2}\xi = \alpha\xi, \tag{5}$$

where ρ_c is the central density, p_c the central pressure, and G the gravitation constant. The central density ρ_c is taken as the polytropic unit of the rest-mass density and the model parameter α as the polytropic unit of length; accordingly, Θ^n is the dimensionless rest-mass density and ξ the dimensionless length.

The integral of hydrostatic Eq. (1) in the Newtonian limit obtains the form ((Fahlman and Anand, 1971), Eq. (12))

$$U = (n+1)\frac{p_c}{\rho_c}\Theta - \frac{1}{2}\Omega^2\alpha^2\xi^2(1-\mu^2) + U_p, \tag{6}$$

where the parameter U_p is the surface gravitational potential at the pole. Alternatively, U_p can be readily incorporated into the potential U (see e.g. (Horedt, 2004), Eq. (4.2.55)).

Next, we introduce the dimensionless perturbation parameters υ and σ ((Fahlman and Anand, 1971), Eq. (15)),

$$\upsilon = \frac{\Omega^2}{2\pi G\rho_c}, \quad \sigma = \frac{1}{c^2}\frac{p_c}{\rho_c}, \tag{7}$$

so-called “rotation parameter”, representing the effects of rotation, and “gravitation parameter” or “relativity parameter”, representing in turn the post-Newtonian effects of gravitation. Accordingly, we can write the generalized Lane-Emden equation in the form (cf. (Fahlman and Anand, 1971), Eqs. (24) and (25))

$$\nabla^2\Theta = -\Theta^n + \upsilon + \sigma\mathcal{E}, \tag{8}$$

where $\mathcal{E}$ is the first-order term in σ ((Fahlman and Anand, 1971), Eq. (30a)); in fact, only the zeroth-order term $\mathcal{E}_0$ in both υ and σ, involved in the right-hand side of this equation, is taken into account here). Terms of order $\mathcal{O}(\sigma\upsilon)$ and $\mathcal{O}(\sigma\upsilon^2)$ (cf. (Fahlman and Anand, 1971), Eq. (25)) are neglected in the present study.

According to PNA, the polytropic density function can be expanded as (cf. (Fahlman and Anand, 1971), Eqs. (26), (35), and (62))

$$\begin{aligned}\Theta(\xi,\mu) &= \sum_{i=0,2}^{4} P_i(\mu)\Theta_i(\xi)\\ &= P_0(\mu)[\alpha_0\theta_{00}(\xi) + \alpha_1\theta_{10}(\xi) + \alpha_2\theta_{20}(\xi) + \alpha_3\theta_{30}(\xi)]\\ &\quad + P_2(\mu)\{\alpha_1 A_{12}\theta_{12}(\xi) + \alpha_2[\theta_{22}(\xi) + A_{22}\theta_{12}(\xi)]\}\\ &\quad + P_4(\mu)\{\alpha_2[\theta_{24}(\xi) + A_{24}\theta_{14}(\xi)]\}\\ &= \sum_{i=0}^{3}\alpha_i\sum_{j=0}^{4}\mathcal{A}_{ij}(\xi)P_j(\mu)\\ &= \alpha_0\theta_{00}(\xi)P_0(\mu)\\ &\quad + \alpha_1[\theta_{10}(\xi)P_0(\mu) + A_{12}\theta_{12}(\xi)P_2(\mu)]\\ &\quad + \alpha_2\{\theta_{20}(\xi)P_0(\mu) + [\theta_{22}(\xi) + A_{22}\theta_{12}(\xi)]P_2(\mu)\\ &\quad + [\theta_{24}(\xi) + A_{24}\theta_{14}(\xi)]P_4(\mu)\}\\ &\quad + \alpha_3\theta_{30}(\xi)P_0(\mu),\end{aligned} \tag{9}$$

where α_i are the perturbation parameters ((Fahlman and Anand, 1971), Eq. (24)); namely, $\alpha_0 = 1$, $\alpha_1 = \upsilon$, $\alpha_2 = \upsilon^2$, $\alpha_3 = \sigma, \alpha_4 = \upsilon\sigma$, and $\alpha_5 = \upsilon^2\sigma$. In the present study, we neglect terms of order $\mathcal{O}(\alpha_4)$ and $\mathcal{O}(\alpha_5)$, as well as terms in Legendre polynomials of degree higher than P_4. The functions Θ_i are readily recognized, e.g.,

$$\Theta_2(\xi) = \alpha_1 A_{12}\theta_{12}(\xi) + \alpha_2[\theta_{22}(\xi) + A_{22}\theta_{12}(\xi)], \tag{10}$$

and likewise the functions $\mathcal{A}_{ij}$, e.g.,

$$\mathcal{A}_{22}(\xi) = \theta_{22}(\xi) + A_{22}\theta_{12}(\xi). \tag{11}$$

The functions θ_{ij} are involved in the differential equations defined by Eqs. (31)–(38) of Fahlman and Anand (1971) with initial conditions given by Eqs. (39)–(41) of Fahlman and Anand (1971). Note that the function θ_{00} coincides with the Lane–Emden function θ of the undistorted Newtonian configuration, since the differential equation (37) in Fahlman and Anand (1971) reduces for $i = 0$ and $j = 0$ to the well-known classical Lane–Emden equation. The parameters A_{ij} ((Fahlman and Anand, 1971), Eq. (59)) multiply properly the homogeneous solutions of θ_{ij} ((Fahlman and Anand, 1971), Eqs. (42), (43)) so that certain boundary conditions be satisfied.

In detail, the functions θ_{ij} obey the differential equations (cf. (Fahlman and Anand, 1971), Eqs. (37) and (38))

$$\frac{d^2\theta_{ij}}{d\xi^2} + \frac{2}{\xi}\frac{\theta_{ij}}{d\xi} - \frac{j(j+1)}{\xi^2}\theta_{ij} = S_{ij}, \tag{12}$$

for $i = 0, 1, 2, 3, j = 0, 2, 4$. Odd js and, accordingly, the respective functions θ_{i1} and θ_{i3} do not appear in this perturbation analysis, since the resulting configuration has to be symmetric under reversal of the direction of rotation. Hence, terms multiplying the odd Legendre polynomials P_1, P_3 must be zero.

The non-trivial functions S_{ij} are given by Fahlman and Anand (1971) (Appendix A),

$$\begin{aligned}S_{00} &= -\theta_{00}^n,\\ S_{10} &= -n\theta_{00}^{n-1}\theta_{10} + 1,\\ S_{1j} &= -n\theta_{00}^{n-1}\theta_{1j},\ j = 2,4,\\ S_{20} &= -n\theta_{00}^{n-1}\theta_{20} - \frac{n(n-1)}{2}\theta_{00}^{n-2}\left(\theta_{10}^2 + \frac{1}{5}A_{12}^2\theta_{12}^2\right),\\ S_{22} &= -n\theta_{00}^{n-1}\theta_{22} - \frac{n(n-1)}{2}\theta_{00}^{n-2}\left[2A_{12}\theta_{12}\left(\theta_{10} + \frac{1}{7}A_{12}\theta_{12}\right)\right],\\ S_{24} &= -n\theta_{00}^{n-1}\theta_{24} - \frac{n(n-1)}{2}\theta_{00}^{n-2}\left(\frac{18}{35}A_{12}^2\theta_{12}^2\right),\\ S_{30} &= -n\theta_{00}^{n-1}\theta_{30} + \frac{\Gamma}{2\Gamma-2}\left(\frac{d^2\theta_{00}^2}{d\xi^2} + \frac{2}{\xi}\frac{d\theta_{00}^2}{d\xi}\right) - 2(n+1)\mathfrak{S}_{30} - \frac{3\Gamma-2}{\Gamma-1}\theta_{00}^{n+1},\end{aligned} \tag{13}$$

where (cf. (Fahlman and Anand, 1971), Eq. (7))

$$\Gamma = \frac{n+1}{n}. \tag{14}$$

According to Fahlman and Anand (1971) (Appendix A), the term $\mathfrak{S}_{30}$ driven by the coefficient $2(n+1)$ in S_{30} has the form

$$\mathfrak{S}_{30} = c_{00}\theta_{00}^n + \theta_{00}^{n+1}. \tag{15}$$

where the constant c_{00} (Fahlman and Anand, 1971, Eqs. (47a) and (48)) is to be determined.

The details on the derivation of the homogeneous-solutions multipliers A_{ij} (cf. (Fahlman and Anand, 1971), Eqs. (42) and (43)) have to do with the continuity of the potential and of its first derivative at the surface of the star (cf. (Fahlman and Anand, 1971), Eq. (52)),

$$U_{\rm surface} = U^e_{\rm surface}, \quad \left(\frac{\partial U}{\partial\xi}\right)_{\rm surface} = \left(\frac{\partial U^e}{\partial\xi}\right)_{\rm surface}. \tag{16}$$