Available online at www.sciencedirect.com

ScienceDirect

New Astronomy 13 (2008) 216–223

New Astronomy

www.elsevier.com/locate/newast

Velocity scaling method to correct individual Kepler energies

Da-Zhu Ma, Xin Wu *, Jun-Fang Zhu

Department of Physics, Nanchang University, Xuefu Road 999, Nanchang, Jiangxi 330031, China

Received 10 July 2007; received in revised form 3 September 2007; accepted 29 September 2007
Available online 6 October 2007
Communicated by W.B. Hubbard

Abstract

As variants of the velocity–position single scaling method of Fukushima, which extends Nacozy's manifold correction scheme to monitor the integrated position and velocity by using the integral invariant relation and the same spatial scale transformation, a new velocity scaling method and a new position scaling method for correcting the varying Kepler energy of each body in an *n*-body problem of planetary dynamics are presented. Compared with Fukushima's idea, the new schemes are simple to operate. Like other existing methods including the method of Fukushima and of Wu et al., the two new methods not only are almost the same effectiveness in significantly improving the orbital semi-major axis or mean anomaly at the epoch, but also can raise the accuracy of numerical integration by several orders. In particular, the new velocity scaling method as well as the method of Wu et al. is the most convenient in application.

PACS: 95.10.Ce; 45.50.Pk; 45.10.−b; 02.70.−c

Keywords: Celestial mechanics; Solar system; Stellar dynamics; Numerical integration; *n*-Body simulations; Manifold correction

1. Introduction

It is very clear that a difference of the Kepler energies in a pure two-body problem causes Lyapunov's instability of the Keplerian orbit (Avdyushev, 2003). This must be nutrient for cultivating various errors of a numerical solution. Fortunately, Nacozy's approach (1971) and Baumgarte's method (1973) seem to be best help in the struggle against Lyapunov's instability by suppressing the growth of the energy error. The so-called stabilizing terms depending on the energy integral are directly added to the numerical solution for the former, while to the set of equations of motion for the latter. Because the former is greatly superior to the latter in some sense (Wu et al., 2006b; Wu and He, 2006), hereafter we focus mainly on Nacozy's approach and its extensions and applications in an *n*-body system.

For a perturbed Kepler problem or a system consisting of multiple celestial bodies, the Kepler energy of each body is no longer a constant. In this sense, the applicability of Nacozy's original method of manifold correction becomes difficult. However, such correction approach of the Kepler energy can still be implemented if an integral invariant relation (Szebehely and Bettis, 1971; Huang and Innanen, 1983) is employed. In detail, one can get a value of the Kepler energy at every integration step by integrating the time derivative of the Kepler energy and the equations of motion simultaneously. Generally, this value of the Kepler energy along this direction is of higher accuracy than one from direct integration of the equations of motion. In other words, it is regarded as to a more precise reference value of energy stabilization. Noting this fact and according to the constraints of a two-body problem, Liu and Liao (1988, 1994) proposed their scaling method, where two distinct scale factors, α and β, are adopted to adjust the position and velocity of a body, respectively. The two factors are connected with the relation $\alpha\beta^2 = 1$. As we have known, the scaling method is the first extension to Nacozy's approach. On the other hand, in the light of a circular orbit integrated by Euler's method Fukushima (2003) found that

* Corresponding author. Fax: +86 791 3969512.
E-mail address: xwu@ncu.edu.cn (X. Wu).

1384-1076/$ - see front matter
doi:10.1016/j.newast.2007.09.002

the integrated position and velocity should be used the same scale factor, which satisfies a certain cubic equation related the Kepler energy. As an illustration, it is necessary to apply Newton's method to solve the cubic equation such that the scale factor can be determined. Fukushima's scaling method is another example of the extension of Nacozy's approach. Recently, Wu et al. (2007) gave a brand-new direct extension of Nacozy's approach. In their method, only three components of the velocity vector are corrected. Namely, it means that only a scale transformation to the velocity is considered. In this case, this correction scheme is also viewed as one of scaling methods. As Wu et al. stated, the above three methods originate from different correction paths to pull the numerical solution of a trajectory back to the energy hypersurface on which it should lie. The two scaling methods of Fukushima and Liu and Liao bring the corrected solution to fall exactly on the energy hypersurface along particular, but not the shortest, directions. As far as the correction approach of Wu et al. is concerned, it goes along the direction of the steepest descent to the hypersurface, while the corrected solution does not fall rigorously on the hypersurface. In spite of these facts, all the three methods are almost effective in the sense of drastic improving the orbit precision in some circumstances. For more information, see the article of Wu et al. (2007).

The examples mentioned above have told us that there is wide freedom in the path to the energy manifold. Really, we find other single scaling methods that will become the focus of attention in the present paper. In the new techniques, we can only use a scale factor to the integrated three-dimensional either velocity or position. It should be pointed out that it is easier to get these scale factors in our current schemes than to do ones of Fukushima's idea (2003). This paper is organized as follows. Section 2 describes two new single scaling methods in detail. Then we shall roughly compare them with other existing correction methods, such as the method of Fukushima (2003) and of Wu et al. (2007). For an in-depth exploration, in Section 3 we mainly check the numerical validity of these methods by some test templates. Finally, the summary follows in Section 4.

2. Several single scaling methods

At first, we use an integral invariant relation (Szebehely and Bettis, 1971; Huang and Innanen, 1983) to determine a reference value of Kepler energy at per integration step. Then, we introduce two new single scaling methods with corrections of the Kepler energy. Finally, the two methods are roughly compared with the method of Fukushima (2003) and of Wu et al. (2007).

2.1. Integral invariant relation

A perturbed one-body problem in the heliocentric coordinate system has its equation of motion in the form

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -\frac{\mu}{r^3}\boldsymbol{r} + \boldsymbol{a}, \tag{1}$$

where $\boldsymbol{r}$, $\boldsymbol{v}$, $\mu \equiv G(M+m)$, $r = |\boldsymbol{r}|$ and $\boldsymbol{a}$ are the position, velocity, gravitational constant, radius and perturbing acceleration, respectively. The Kepler energy of this body is defined as

$$K = \boldsymbol{v}^2/2 - \mu/r. \tag{2}$$

For the existence of the perturbation, K becomes a function of time such that

$$\frac{\mathrm{d}\Delta K}{\mathrm{d}t} = \boldsymbol{v}\cdot\boldsymbol{a} \tag{3}$$

with $\Delta K = K - K_0$, where K_0 stands for the starting value of the Kepler energy. Eq. (3) is called as an integral invariant relation (Szebehely and Bettis, 1971; Huang and Innanen, 1983) with respect to the Kepler energy.

From the analytical point of view, the value of K given by Eq. (2) and one derived from Eq. (3) have no any difference. However, they differ greatly from the numerical point of view. The reason is that the latter K deals with the magnitude of the perturbation acceleration $\boldsymbol{a}$, and should have a more precision in numerical integration. For a further illustration, we use a certain numerical scheme[1] to integrate Eq. (1) in order to get a numerical solution $(\boldsymbol{r}_I, \boldsymbol{v}_I)$. Substituting the numerical solution into Eq. (2), we have the value of K, labeled as K_I. On the other hand, we adopt still this integrator to work out Eqs. (1) and (3), and get the value of K, marked as K^*. As an emphasis, we would rather integrate Eq. (3) than the equation $\mathrm{d}K/\mathrm{d}t = \boldsymbol{v}\cdot\boldsymbol{a}$ so that the accumulation of round-off errors can be reduced significantly (Fukushima, 1996). In terms of the statement above, K^* is closer to the actual value of the Kepler energy than K_I. Therefore, K^* is regarded as to the reference value of the Kepler energy. This gives a good chance to adjust the numerical solution $(\boldsymbol{r}_I, \boldsymbol{v}_I)$ to a more accurate solution $(\boldsymbol{r}^*, \boldsymbol{v}^*)$ so that the corrected K_I is exactly equal to K^*. Now a problem is how to obtain $(\boldsymbol{r}^*, \boldsymbol{v}^*)$ by means of known K^* and $(\boldsymbol{r}_I, \boldsymbol{v}_I)$. In principle, there are infinite relations between $(\boldsymbol{r}_I, \boldsymbol{v}_I)$ and $(\boldsymbol{r}^*, \boldsymbol{v}^*)$ which we are permitted to construct. That is to say, there are many and many possible manifold corrections of the Kepler energy. In particular, scale transformations may be the simplest of various manifold correction methods. The scaling methods of Liu and Liao (1988, 1994) and Fukushima (2003) are some successful examples. Without question, there should be other scaling methods. We will deliver ourself of our findings in the following.

2.2. New single scaling methods

Based on Eq. (2), K^* and $(\boldsymbol{r}^*, \boldsymbol{v}^*)$ should satisfy the constraint

[1] Here a symplect integrator (e.g. Wu et al., 2003), with the preservation of symplectic structure and an integral of energy, should be excluded.