New Astronomy 14 (2009) 607–614

Contents lists available at ScienceDirect

New Astronomy

journal homepage: www.elsevier.com/locate/newast

Universal formulation of quasi-Keplerian motion, and its applications

Christian Hellström *, Seppo Mikkola

Tuorla Observatory, Department of Physics and Astronomy University of Turku, Väisäläntie 20, 21500 Piikkiö, Finland

ARTICLE INFO

Article history:
Received 16 February 2009
Accepted 4 March 2009
Available online 17 March 2009
Communicated by E.P.J. van den Heuvel

PACS:
91.10.Sp
95.10.Ce
05.45.−a
02.60.Cb

Keywords:
Kepler–Coulomb
Universal variables
Precession
J_2

ABSTRACT

We derive the universal solution to the Kepler–Coulomb problem with an additional inverse-square potential, valid for any type of orbit, and describe three prominent applications in astrodynamics: the relativistic precession of the apsides, the numerical integration of perturbed Kepler–Coulomb problems with a generalized leapfrog, and the averaged motion of earth-orbiting satellites with the J_2 perturbation. The modified orbital elements and Delaunay variables are presented as well.

1. Introduction

In the class of dynamical systems with central force fields, the Kepler–Coulomb problem is perhaps the most famous. The Kepler–Coulomb problem in three-dimensional Euclidean space is related to the harmonic oscillator in four dimensions through the Kustaanheimo–Stiefel transformation (Kustaanheimo and Stiefel, 1965), or the Duru–Kleinert transformation (Duru and Kleinert, 1979), as it is known in the context of atomic path integrals, which linearizes and regularizes the equations of motion. In celestial mechanics, the solution of the Kepler–Coulomb problem is often written in terms of a universal variable based on the Sundman transformation; the solution is valid irrespective of the orbit type.

The Hamilton–Jacobi equation for the Kepler–Coulomb problem is known to be separable in four coordinate systems only: spherical, parabolic, elliptic and spheroconical (Cordani, 2003). Adding a generic perturbation generally destroys the integrability and symmetry of the original system. However, McIntosh and Cisneros (1970), and Zwanziger (1968), who focussed solely on its quantum mechanical implications, discovered that the added potential of a self-dual Dirac (magnetic) monopole preserves the symmetries of the Kepler–Coulomb problem, and hence that it remains integrable. The MICZ problem has been solved formally by Bates (1988) using Souriau's regularization technique (see e.g. Cordani, 2003, Chapter 5). Although Bates' solution is simple and elegant, it lacks a clear physical interpretation; third-order derivatives of the coordinates with respect to the (reparameterized) time are required.

In this paper, we review the classical MICZ problem from a universal variable point of view, as outlined previously by Caballero and Elipe (2001), and we address its relevance to astrodynamics. In Section 2, we briefly review the symmetries of the classical MICZ problem in relation to the Kepler–Coulomb problem. The similarities between these systems enable us to write down an analytical solution that is valid for any type of orbit, as shown in Section 3. Following Deprit (1981), the MICZ problem is commonly denominated in astrodynamical literature quasi-Keplerian for reasons that will become apparent below. Section 4 comprises a discussion of the quasi-Keplerian orbital elements and Delaunay variables, from which other orbital representations may be obtained. Three applications are described in detail in Section 5. These include the in-plane precession of Keplerian orbits due to the lowest-order correction due to the general theory of relativity; the numerical integration of perturbed Kepler–Coulomb systems by means of time transformations that split the total Hamiltonian into the quasi-Keplerian Hamiltonian and the perturbation; and the secular J_2 effect, which is relevant to all satellites and spacecraft in low orbits around an oblate primary, such as the Earth. We have endeavoured to maintain the discussion general, so that our results can be applied to related physical systems, such as for example Rydberg atoms in the presence of magnetic monopoles.

* Corresponding author. Fax: +358 243 3767.
E-mail address: christian.hellstrom@utu.fi (C. Hellström).

1384-1076/$ - see front matter
doi:10.1016/j.newast.2009.03.001

2. MICZ problem

The Hamiltonian of the McIntosh–Cisneros–Zwanziger (MICZ) system is

$$H(\boldsymbol{q},\boldsymbol{p}) = \frac{1}{2}p^2 - \frac{k}{q} + \frac{\mu^2}{2q^2}, \tag{1}$$

where $k \in \mathbb{R}$, $\boldsymbol{p}$ and $\boldsymbol{q}$ are the canonical momenta and coordinates, respectively, and $p = |\boldsymbol{p}|$ and $q = |\boldsymbol{q}|$, the standard Euclidean (ℓ^2) norm. The monopole potential is actually derived from a vector potential $\boldsymbol{A}$, such that the magnetic field $\boldsymbol{B} = \nabla \wedge \boldsymbol{A} = \mu\boldsymbol{q}/q^3$ for $\boldsymbol{q} \in \mathbb{R}^3 \setminus \{0\}$. This in turn means that in going from the Kepler–Coulomb to the MICZ problem, the canonical momenta are transformed through a minimal substitution: $\boldsymbol{p} \mapsto \boldsymbol{p} - \boldsymbol{A}$. The parameter μ can be viewed as a deformation parameter; for $\mu = 0$ one recovers the well-known Kepler–Coulomb problem. The associated one-parameter family of differential equations is Hamiltonian at each value of μ, as shown by Bates (1988). These Hamiltonian deformations are distinct for different values of the deformation parameter. We shall require that $\mu^2 \in \mathbb{R}$ in order to allow for both positive and negative (real) values.

The phase space of the classical MICZ problem (1), $T^\star(\mathbb{R}^3 \setminus \{\boldsymbol{0}\})$, is endowed with the Poisson structure

$$\{q_i, q_j\} = 0, \quad \{p_i, q_j\} = \delta_{ij}, \quad \{p_i, p_j\} = -\mu \sum_{k=1}^{3} \epsilon_{ijk} q_k / q^3, \tag{2}$$

and the herewith related canonical two-form

$$\omega = \sum_{i,j=1}^{3} \delta_{ij}\,\mathrm{d}p_i \wedge \mathrm{d}q_j - \frac{\mu}{2q^3} \sum_{i,j,k=1}^{3} \epsilon_{ijk} q_i\,\mathrm{d}q_j \wedge \mathrm{d}q_k. \tag{3}$$

The first term in Eq. (3) is simply the symplectic two-form of the Kepler–Coulomb system. The additional dynamics due to the monopole potential are incorporated in the second term. Physically, μ is the magnetic 'charge' of the monopole. Mathematically, the deformation parameter can be interpreted as the De Rham cohomology class of the symplectic form.

Since the Hamiltonian function is independent of time, it is a conserved quantity. A simple calculation reveals that because of rotational invariance,

$$\boldsymbol{J} = \boldsymbol{q} \wedge \boldsymbol{p} - \mu \frac{\boldsymbol{q}}{q} \tag{4}$$

is conserved as well. In analogy with the Kepler–Coulomb problem, there is an additional conserved quantity:

$$\boldsymbol{R} = \frac{1}{\sqrt{2|H|}} \left(\boldsymbol{p} \wedge \boldsymbol{J} - \frac{\boldsymbol{q}}{q} \right), \tag{5}$$

which corresponds to the Laplace–Runge–Lenz vector. Under the Poisson bracket the Laplace–Runge–Lenz vector yields the $\mathfrak{so}(4)$ algebra for $H < 0$, that is, the Lie algebra of the four-dimensional rotation group, and $\mathfrak{so}(3,1)$ for $H > 0$, which is the Lie algebra of the conformal group in four-dimensions. These are the same symmetries of the Kepler–Coulomb problem. In fact,

$$(\boldsymbol{J}, \boldsymbol{q}/q) = -\mu, \tag{6}$$

which shows that the solutions to Hamilton's equations for Eq. (1) are conic sections, where the opening angle $\vartheta = \arccos \mu/J$. Similarly, we have that

$$(\boldsymbol{p} \wedge \boldsymbol{J}, \boldsymbol{q}) = \sqrt{2\mid H\mid}(\boldsymbol{R}, \boldsymbol{q}) - q = J^2 - \mu^2, \tag{7}$$

so that the motion lies in the plane.

3. Universal solution

Because the motion is planar, it is convenient to use polar coordinates $\boldsymbol{q} = (r, \theta)$ in the orbital plane:

$$H = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) - \frac{k}{r} + \frac{\mu^2}{2r^2}. \tag{8}$$

The time evolution of the system is encoded in Hamilton's equations:

$$\dot{q}_k = \frac{\partial H}{\partial p_k}, \quad \dot{p}_k = -\frac{\partial H}{\partial q_k}, \tag{9}$$

where the overdot represents the derivative with respect to the time t. In polar coordinates, Hamilton's equations for the Hamiltonian (8) simply read

$$\dot{r} = p_r, \quad \dot{p}_r = \frac{p_\theta^2 + \mu^2}{r^3} - \frac{k}{r^2}, \tag{10}$$

$$\dot{\theta} = \frac{p_\theta}{r^2}, \quad \dot{p}_\theta = 0. \tag{11}$$

It is customary to reparameterize the time t with a so-called Sundman transformation

$$\mathrm{d}t = r\,\mathrm{d}s. \tag{12}$$

Let the prime ($'$) denote the derivative with respect to the independent (universal) variable s, that is, $f'(s) = \mathrm{d}f(s)/\mathrm{d}s = r\dot{f}(t)$ for any $f \in \mathscr{C}^1(\mathbb{R}, \mathbb{R})$ by virtue of the chain rule and the transformation (12). Therefore, we find that

$$\begin{aligned} r'' &= (rp_r)' \\ &= r\frac{\mathrm{d}(rp_r)}{\mathrm{d}t} \\ &= r\left(p_r^2 + \frac{p_\theta^2 + \mu^2}{r^2} - \frac{k}{r}\right). \end{aligned} \tag{13}$$

Since the Hamiltonian is constant, we have the identity

$$2H_0 - \left(p_r^2 + \frac{p_\theta^2 + \mu^2}{r^2} - \frac{2k}{r}\right) = 0, \tag{14}$$

where H_0 is the initial (numerical) value of the Hamiltonian. Now, we can add this expression to the bracketed expression of Eq. (13) and obtain

$$r'' = 2H_0 r + k. \tag{15}$$

Hence, we arrive at the set of differential equations:

$$\begin{cases} r'' = 2H_0 r + k, \\ \theta' = p_\theta / r, \\ t' = r, \end{cases} \tag{16}$$

which are to be supplied with initial conditions. The deformation parameter μ does not appear explicitly in these equations; it only resides in the value H_0. Let r_0, r_0' and r_0'' denote the initial values of r, $r' = (\boldsymbol{q}, \boldsymbol{p})$ and $r'' = 2H_0 r + k$, respectively. The system (16) is then easily solved in terms of Stumpff functions by

$$\begin{cases} r(s) = r_0 + r_0' s c_1(-2H_0 s^2) + r_0'' s^2 c_2(-2H_0 s^2), \\ \theta(s) = p_\theta \xi(s), \\ t(s) = r_0 s + r_0' s^2 c_2(-2H_0 s^2) + r_0'' s^3 c_3(-2H_0 s^2), \end{cases} \tag{17}$$

where we have defined

$$\xi(s) = \int_0^s \frac{\mathrm{d}\varphi}{r(\varphi)}. \tag{18}$$

Using the universal Stumpff functions and the definition $\beta = -2H_0$, the solution to $r(s)$ and $t(s)$ may be written more concisely as