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Evolution of some quasicircular orbits in the planetary problem

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A R T I C L E   I N F O

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A B S T R A C T

We derive in the plane problem a new closed solution of the Lagrangian equations for resonant motion, concomitantly including zeroth order, and approximate first order and secular perturbations. A major aim is the determination of simple lower limits for the maximum eccentricities and variations of semimajor axes (intrinsic values). Applications of the general solution are made for each perturbation separately. (i) Zeroth and first order perturbations: a new closed solution for the principal zeroth order variation of semimajor axes is obtained. The maximum eccentricity and relative change of semimajor axis of any lunar orbit cannot be lower than 0.019 and 0.018, respectively. (ii) Secular perturbations: with the angular momentum integral our secular perturbations can be easily extended to the spatial problem. The planetary Lidov–Kozai problem is extended to retrograde orbits, showing that large variations of eccentricity and inclination occur for initially circular orbits, if initial mutual inclinations are between about 40° and 150°. (iii) Resonant perturbations: for first and second order resonances, and initially circular orbits our formulas generally approximate just the calculated orbital elements during the whole motion. As a new unexpected result, the numerical exploration of the asteroid belt reproduces most of its overall characteristics up to third order resonances within the restricted three-body problem and modest initial eccentricities ≤ 0.05.

1. Introduction

This paper deals with the orbital evolution of two small masses, orbiting in a plane round a much larger mass in initially nearly circular, sufficiently separated orbits. What are the outcomes of initially quasicircular orbits? Circular orbits represent a natural limit of elliptic motion, and it is commonly believed that their study has marginal interest, since the orbits will stay quasicircular. Contrary to this belief and anticipating some results, I find in the next sections that (i) the maximum eccentricity and variation of semimajor axis of initially circular, nonresonant orbits are unexpectedly large for the *distant* outer mass (Section 3), (ii) inclinations and eccentricities exhibit large variations for distant, initially circular, roughly perpendicular planetary orbits (Section 4), (iii) maximum variations of semimajor axes and eccentricities of first and second order resonances of initially circular orbits are appreciable in the plane problem, but severely bounded by simple analytical formulas [Eqs. (61) and (62), Fig. 9 bottom].

The results of this paper have direct applicability to the following topics:

(i) Planetary and satellite systems, especially if only few observations are available. If initial values of the orbital invariants (48)–(50) are known, they often provide meaningful bounds for the eccentricity e_1 and inclination I_1 of a nonresonant (extrasolar) planet, after setting the elements of the other planets equal to extreme values $e_i = 0, 1$ and $I_i = 0°, 180°$. And Table 1 shows that the large observed eccentricities reported by Tokovinin (2001, p. 89; 2004, p. 12) for the inner orbit in triple (multiple) stellar systems occur in planetary systems too for initially circular, roughly perpendicular orbits. So, the existence of an exterior perturbing planet in a roughly perpendicular orbit could be an explanation (among many others) for the numerous observed large eccentricities of extrasolar planets (Marcy et al., 2000, p. 1304; Takeda et al., 2009).
Because *secular* variations of semimajor axes occur only at third order in the disturbing masses (e.g. Stumpff, 1974, p. 368), it seems important to have a simple closed formula for the principal, periodic nonresonant variation of semimajor axis in planetary and satellite systems, as given by Eqs. (37)–(44).

(ii) Planetary rings. As emphasized by Franklin et al. (1984, p. 566), incorporation of resonant perturbations between a ring particle and a neighbouring satellite can complement the density wave theory of planetary rings having negligible self-gravitation. Thus, our simple formulas from Eqs. (55)–(62) are suitable for the calculation of the maximum resonant changes of eccentricity and semimajor axis of a ring particle orbiting at the inner or outer corotation eccentricity resonance and at the inner or outer eccentric (Lindblad) resonance. Actually, some of the gaps, edges, and arcs

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Table 1
Maximum relative variation of semimajor axes $\Delta a_i/a_i = 2(a_{i,max} - a_{i,min})/(a_{i,max} + a_{i,min})$ $(i = 1,2)$, maximum eccentricity $e_{i,max}$, maximum and minimum inclination $I_{i,max}, I_{i,min}$ with respect to the plane $I_{10} = 0^\circ$, and maximum relative variation $\Delta C/C$, $\Delta C_I/C_I$ of the secular constants C, C_I from Eqs. (47) and (48) for initially circular orbits ($e_{i0} = 0$) of two equal masses $m_i/M = 0.001$ started with $\alpha = a_1/a_2 = 0.3$ and selected values of the initial inclination I_{20} of m_2. Note, the strong variation of $e_{i,max}$ at 41–42° and 148–149° due to the Lidov–Kozai mechanism.

I_{20}	$\Delta a_1/a_1$	$\Delta a_2/a_2$	e_{1max}	e_{2max}	$\Delta C/C$
41°	0.0001	0.008	0.003	0.007	0.003
42°	0.0001	0.009	0.120	0.017	0.003
60°	0.0003	0.016	0.619	0.119	0.005
90°	0.0006	0.041	0.947	0.301	0.024
120°	0.0003	0.058	0.973	0.253	0.227
148°	0.0001	0.008	0.182	0.035	0.011
149°	0.0001	0.006	0.006	0.008	0.010
179°	0.0001	0.006	0.002	0.009	0.007

I_{20}	I_{1min}	I_{1max}	I_{2min}	I_{2max}	$\Delta C_I/C_I$
41°	0°	53°	12°	41°	0.003
42°	0°	55°	13°	42°	0.006
60°	0°	79°	19°	60°	0.101
90°	0°	116°	37°	90°	0.598
120°	0°	177°	57°	120°	–
148°	0°	121°	91°	148°	0.034
149°	0°	118°	93°	149°	0.010
179°	0°	4°	177°	179°	0.007

occurring in planetary rings can be explained in terms of these resonances with observed satellites (mainly first order resonances to be discussed in Appendix A): Mimas, Janus, Pandora, and Prometheus for Saturn's A and B ring, Cordelia and Ophelia for the ϵ ring of Uranus, and Galatea for the arcs in Neptune's Adams ring (Murray and Dermott, 1999, Chapter 10).

(iii) The asteroid and Edgeworth–Kuiper belt. Our simple formulas from Eqs. (55)–(62), (A.3), (B.7), together with the resonance coefficients from Table 3 allow for a quick calculation by hand of the maximum variations of eccentricity and semimajor axis for resonant asteroids and Edgeworth–Kuiper objects. Specifically, an asteroid at maximum 2:1, 3:1, 5:3 resonance with Jupiter must have a maximum eccentricity larger than 0.200, 0.078, 0.074, respectively, while a Plutino at maximum 2:3 resonance with Neptune possesses maximum eccentricity > 0.053, in accordance with the observed deficiency of Plutinos with eccentricity < 0.1 (Chiang et al., 2007; Gomes, 2009, Fig. 1).

Hence, a main concern of this paper is to derive in the plane problem simple analytical *minima* for the *maximum* eccentricities and variations of semimajor axes, occurring always for initially nearly circular orbits. And for a given initial ratio of semimajor axes these lower limits, referred to as intrinsic values, cannot be diminished further by any choice of initial conditions.

The plan of the paper is as follows. In Section 2 we derive a new, exact closed solution of the Lagrangian equations for plane resonant motion, concomitantly including zeroth order, and approximate first order and secular perturbations.

In Section 3 we show that analytical perturbations of semimajor axis and eccentricity up to first order are sufficient to describe the plane motion of initially circular orbits, excepting for very close, resonant, and some distant orbits (Figs. 1–3). We also derive a new, closed analytical formula for the intrinsic variation of semimajor axes (Fig. 1). A brief application to the Moon shows that any lunar orbit has a maximum eccentricity > 0.019 and a relative variation of semimajor axis > 0.018 (Fig. 4).

In Section 4 we extend the classical secular invariants to pro- and retrograde orbits, large eccentricities and inclinations, with particular emphasis on the planetary Lidov–Kozai problem (Libert and Henrard, 2007; Farago and Laskar, 2010).

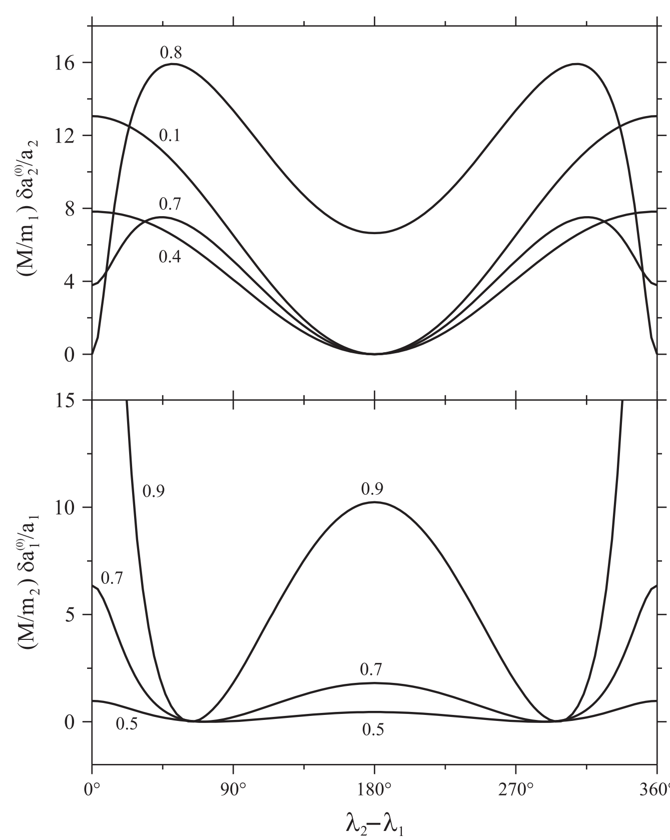

Fig. 1. Theoretical zeroth order variation of semimajor axes $\delta a_i^{(0)}/a_i = (a_i^{(0)} - a_{i,min}^{(0)})/a_i$ $(i = 1,2)$ for quasicircular orbits as a function of mutual mean longitude difference $\chi = \lambda_2 - \lambda_1$, according to Eqs. (41) and (42) if $m_i \ll M$. Values of $\alpha = a_1/a_2$ are indicated on the curves.

Sections 5 and 6 deal with the analytical and numerical outcomes of resonant motion. In Section 5 we obtain the maximum variations of eccentricity and the maximum libration width for *individual* resonances of arbitrary order, including uniform and resonant pericentre motion. For first and second order resonances and initially circular, sufficiently distant orbits, Eqs. (61) and (62)