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Hydrostatic, isothermal planetary models from the point of view of dynamical systems: "Determination of explicit formulas for the point of critical mass"

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Abstract

We investigate a planetary model in spherical symmetry, which consists of a solid core and an envelope of ideal and isothermal gas, embedded in a gaseous nebula. The model equations describe equilibrium states of the envelope. So far, no analytical expressions for their solutions exist, but of course, numerical results have been computed. The point of critical mass, above which no more static solutions for the envelope exist, could not be determined analytically until now. We derive explicit formulas for the core mass and the gas density at the core surface, for the point of critical mass. The critical core mass is also an indicator for the ability of a core to keep its envelope when the surrounding nebula is removed, because at this point, the core's influence extends up to the outer boundary at the Hill radius.

Keywords: Planet formation; Solar system; Autonomous dynamical systems; Point transformations

0. Introduction

In recent years, planet formation has raised a lot of new questions due to observational advances like the discovery of extra-solar gas giants. In the nucleated instability hypothesis, gas envelopes are formed around planetary cores, which in turn are built up by accretion of km-sized solid bodies. The envelope structure has been investigated with hydrostatic models, e.g. by Ikoma et al. (2001), Mizuno (1980), Papaloizou and Terquem (1999), Stevenson (1982), Wuchterl (1993). There was found an upper mass limit for static envelopes—the critical mass—above which no static solutions exist for given nebula conditions. The dependence of this critical mass on nebula conditions or material properties (opacities) is a quite complex problem. In the isothermal idealization used in this article, we can make an analytical approach, giving us deeper insights into the basic features of core–envelope structures and especially the critical mass.

As there are several definitions of the critical mass and the respective critical core mass (CCM) (cf. Wuchterl, 1991), we have to specify what we mean with "critical mass". We follow the definition introduced by Mizuno (1980) where the CCM is defined as the first maximum of $M_c(M_{tot})$. For a given total mass M_{tot} an eigenvalue problem is solved, giving the respective core mass M_c. Out of a series of increasing total masses a relation $M_c(M_{tot})$ arises. If this relation has a maximum, a contradiction occurs. Seen as a quasi-static time evolution, the core mass is a constantly growing structure but the maximum shows that there is no static solution for a core mass above the maximum, so this core mass is called the critical one after Mizuno (1980), hereafter called "classical" CCM. This definition makes no statements about the further evolution of a critical configuration, only the static possibilities are ruled out. It should be noted that the described procedure to get the CCM is done for fixed nebula parameters, i.e. for given outer density and temperature. This fact leads to another description of the phenomenon, also used in this article (cf. Fig. 1), presenting a manifold of possible static core–envelope structures in a parameter set of core mass

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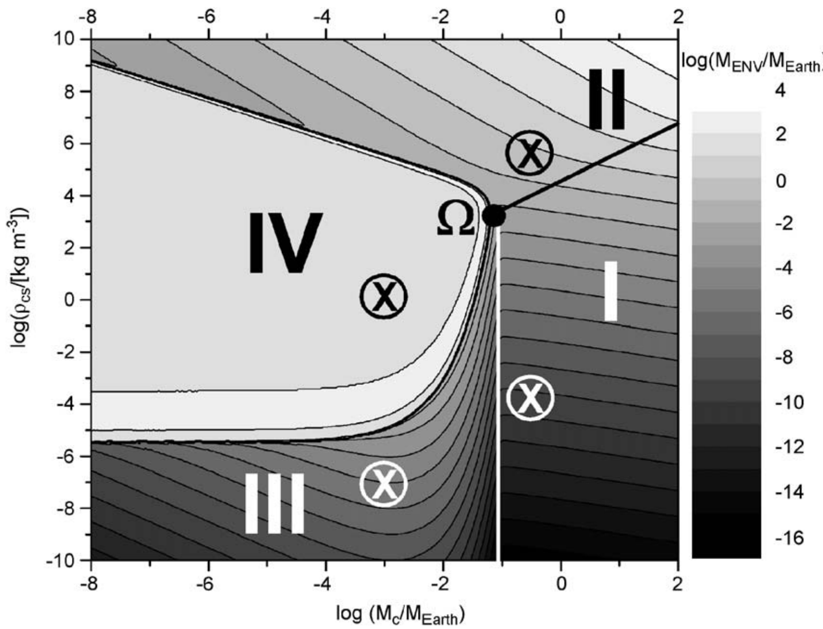

Fig. 1. Contour plot of envelope mass at the Hill radius, function of core mass M_c and gas density ρ_{cs} at core surface. Parameters are given in the text.

M_c and gas density (or pressure) at the core surface ρ_{cs} (Broeg, 2006; Pečnik and Wuchterl, 2005). In this description we find the point of CCM if we fix the outer temperature and follow a line of constant outer density in the direction of increasing total mass. We will reach at least one maximum in the core mass, corresponding exactly to the classical CCM. A nice visualization how the $M_c(M_{tot})$-plot can be understood as a projection out of the manifold is shown in Broeg (2006). It shows that the CCM is closely related to the highest core mass value of the so-called ''region IV'' in the described parameter set, our point Ω in Fig. 1. The direct connection to our work gives the isothermal investigation by Pečnik and Wuchterl (2005). Besides the local CCM, defined there as the outer density depending classical CCM, they introduce a global CCM which is independent of the outer density and corresponds again to our point Ω. In fact this global CCM is the classical CCM for the highest possible outer density permitting hydrostatic equilibrium. The finding that the classical CCM is quite independent of the outer density (Mizuno, 1980) can be explained with the strong decrease of the outer density for increasing core mass beyond the global CCM, the point Ω, because of the strong gravitational influence the core gets. So the global and classical CCMs do not differ significantly since all outer densities are realized in a short core mass range. But this behaviour changes for small orbital distances (Ikoma et al., 2001) or fully convective envelopes (Wuchterl, 1993). For small orbital distances this happens because the above-mentioned fast drop in outer density beyond the point Ω is slowed down due to increasing gas temperature. Naturally the global CCM has the smallest value under all possible CCMs with their respective nebula densities, since it is related to the highest outer density.

Being a main assumption of this article, the isothermal approximation has to be discussed. As all attempts to justify this assumption involve a lot of vague estimations about opacity, luminosity and other temperature gradient related parameter values (e.g. Pečnik and Wuchterl, 2005), we choose a rather pragmatic way to show the applicability of the isothermal assumption. Clearly, for high densities especially in the compact near core parts of the envelope, significant temperature gradients appear. But in the parameter range of interest for this work, i.e. for densities up to $\rho \approx 1 \cdot 10^3\,\mathrm{kg\,m^{-3}}$ (cf. the ''height'' of the point Ω in Fig. 1), we justify the approximation by comparing our results with respective non-isothermal investigations. Critical density (cf. Section 4.1) as well as the global CCM (cf. Section 4.6) agrees up to a factor of 2 with the respective non-isothermal results. So the isothermal investigation provides more than just qualitative answers.

For a review of the whole topic cf. Wuchterl et al. (2000), for a detailed discussion of the critical mass cf. Wuchterl (1991).

1. The model and its equations

Our model contains a solid, rigid core, with mass M_c and mean density ϱ_c, so that we get the core radius

$$r_c = \sqrt[3]{\frac{3M_c}{4\pi\varrho_c}}.$$

The envelope starts at r_c with a given density $\rho(r_c) := \rho_{cs}$ (gas density at the core surface). To describe the mass distribution $M(r)$ and density profile $\rho(r)$ in the envelope, we need the following equations. The force density balance