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Quantization of a uniform gravitational field with a damping quadratic force

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ABSTRACT

A quadratic damped mechanical system is quantized using the WKB formalism. In particular, a uniform gravitational field with a damping force proportional to the square of its speed has been treated in this paper. We construct the Hamiltonian of the system and use it to obtain the Hamilton-Jacobi equation. The action function S has been found using the separation of variables and the chain rule. This function enables us to formulate the wave function and to quantize the system using the WKB approximation. Finally, we are able to prove that the quantum results are in exact agreement with the classical results.

1. Introduction

The presence of damping in a mechanical system is a natural occurrence, most of the processes take place in nature are dissipative processes. This topic gained focus in the last few decades, where many different studies have been done to try to write the Lagrangian and the Hamiltonian that leads to the same equations of motion produced by Newton's mechanics [1–3]. Once you are able to construct the Hamiltonian of any system, you will be able to use the suitable quantization technique to transfer your system to the quantum level [4,5]. the most preliminary study of the dissipative systems were done by Bauer [6] when he tried to prove that the variational principle does not succeed in the case of the dissipative linear systems. Bauer's argument faced a number of problems, which were handled by Bateman later on [7]. Bateman added a new exponential factor to the regular Lagrangian, which represents the decaying factor, and the his formulation for the Lagrangian and the constructed Hamiltonian were able to produce the same equations of motion.

The problem of constructing a quantum theory for the damped dissipative systems, e.g. motion in gravitational field under a damping effect, has attracted attention for more than 50 years. In spite of the success of "interaction with reservoir" approaches, we feel there is still room for some formal developments in this direction. In this paper we analyze a novel Lagrangian model for the motion of an object in a gravitational field under a damping effect depending on the square of its speed, which is nothing but an extension of the work done in Ref. [8].

In this paper, we extend the work done in Ref. [9] recently by including systems under damping effect proportional to the square of their velocity and we investigate our systems using the Hamilton-Jacobi equation (HJE) [10]. After constructing the Hamiltonian of the system and writing the corresponding o(*HJE*), the separation-of-variables technique has been used to find the corresponding principal function. The equations of motion could then be derived from this function, which represents the

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energy of the system, in terms of the generalized coordinates and momenta. This, in turn, could constitute a basis for the so-called canonical quantization using the WKB approximation, thereby obtaining the corresponding Hamiltonian and Schrödinger's equation. In this paper we show an introduction about the topic in Section 1, while in Section 2, we remind the reader of the Hamilton-Jacobi formulation and Section 3 contains WKB quantization. Our example of the quadratic damping effect is presented in Section 4 and finally we conclude our results.

2. Hamilton-Jacobi formulation

The Lagrangian formulation of classical system requires that the system is formulated by n generalized coordinates q_i and velocities $\dot{q}_i$, and generalized velocity q_i, according to that we can write the Lagrangian as a function of these general coordinates $L(q_i, q_i^{\cdot})$ [11]. In case of dissipative systems the corresponding system can take the form

$$L_o = L(q_i, q_i^{\cdot})e^{\lambda t} \tag{1}$$

and the corresponding Euler equations are given by

$$\frac{\partial L_o}{\partial t} - \frac{d}{dt}\left(\frac{\partial L_o}{\partial q_i^{\cdot}}\right) = 0 \tag{2}$$

According to that the canonical Hamiltonian can be written as

$$H_o = \sum p_i \dot{q}_i - L_o \tag{3}$$

where, the momenta in the system can be obtained from

$$p_i = \frac{\partial L_o}{\partial q_i^{\cdot}} \tag{4}$$

As a result of that, and once the Hamiltonian is constructed in the system, one can write the Hamilton Jacobi Equation (HJE) as [11].

$$H_o\left(q_1, \ldots, q_n, \frac{\partial S}{\partial q_1}, \ldots., \frac{\partial S}{\partial q_n}\right) + \frac{\partial S}{\partial t} = 0 \tag{5}$$

where $p_i = \frac{\partial S}{\partial q_i}$. It is known in standard formulation that the Hamiltonian-Jacobi problem focuses on finding $S(q, t)$, which satisfies Eq. (5). One can achieve that by using the method of separation of variables, by assuming $S(q, \alpha, t) = \eta(q, \alpha) - \alpha t$, where α is a constant, and $\eta(q, \alpha)$ is time independent and called Hamilton's characteristic function, it follows that

$$H\left(q, \frac{\partial \eta}{\partial q} = \alpha\right) \tag{6}$$

where $\frac{\partial S}{\partial \alpha} = \beta_i$ For the purpose of quantizing our systems one can use WKB approximation which is WKB approximation is a technique for simplifying differential equations using a number of assumptions [12–14]. In the context of quantum mechanics, the WKB method can be used to simplify the Schrödinger equation. In WKB quantization technique, the meaning of the state function ψ is connected in sophisticated way with the Hamilton's principle function S, where using this technique one can write the solution of the Schrödinger equation as

$$\psi(q, t) = Aexp\left[\frac{iS(q, t)}{\hbar}\right] \tag{7}$$

3. WKB quantization

First we shall show how quantum mechanics reproduces the Hamilton-Jacobi equations of classical mechanics in the semi-classical limit $\hbar \to 0$. The Schrödinger equation for a single particle in a potential $V(q)$ is

$$i\hbar \frac{\partial \psi(q, t)}{\partial t} = \left[-\frac{\hbar^2}{2}\left(\frac{\partial^2}{\partial q^2}\right) + V(q)\right]\psi(q, t) \tag{8}$$

Using the substitution