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Principles to construct disjunctive cuts☆

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Abstract

This article focuses on solving the disjunctive problem. Various methods of constructing disjunctive cuts (DC) from the logical limitations on linear inequalities have been presented. A general principle of DC and a principle allowing to strengthen these cuts were established. By virtue of the stated principles, solving the problems of optimization with a great number of limitations can be simplified. Two theorems were formulated and proved. Four examples illustrated various theoretical statements.

The suggested principles and the procedures based on them provide the theoretical background to the elaboration of algorithms intended for software implementation in solving practical problems.

Keywords: Mathematical logic; Disjunctive problem; Convex cover; Set closure; Cut strengthening.

Introduction

Disjunctive methods allow obtaining disjunctive cuts (DCs) from logical constraints on linear inequalities. The present study has examined the principle of DC that summarizes the available methods for constructing the cuts. However, as this principle is connected to other principles and approaches relevant for finding the cuts, it is often called the basic one. The so-called Cut Strengthening Metaprinciple (CSM) is used for strengthening the DC principle.

The DCs discussed in this paper extend the possibilities for obtaining them when constructing new algorithms. The software implementing these algorithms is designed for numerically solving theoretical and practical optimization problems with logical constraints on linear inequalities. The results obtained are already being used in the analysis of the economic models described using logical constraints. Other applications of these results may include, for example, the numerical solution of optimization problems. The theoretical propositions of varying degrees of complexity have been illustrated in this paper by four examples.

The key definitions

Let us examine a disjunctive minimization problem formulated as

$$\min\left\{ cx \mid \bigvee_{i=1}^{m}\left(\sum_{j=1}^{n} a_{i,j}x_j \geq b_i,\ x \geq 0\right) = \text{true}\right\},$$

where $c = (c_1, c_2, \ldots, c_n)$, $x = (x_1, x_2, \ldots, x_n)^T$.

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From now on we shall call it the Disjunctive Program (DP). The expression in the parentheses of the formula is a logical variable that takes a true value if the x point satisfies the system of $n+1$ inequalities (or a false one if it does not).

The subject of inquiry is the admissible set of the disjunctive minimization problem under consideration

$$S_4 = \left\{ x| \bigvee_{i=1}^{m} \left(\sum_{j=1}^{n} a_{i,j}x_j \geq b_i,\ x \geq 0 \right) = \text{true} \right\}.$$

The S_4 set is called disjunctive, and it can be represented in two ways. In the first case its representation is a disjunctive normal form of a logical expression containing linear inequalities (any such expression can be reduced to the disjunctive form). The second way of representing a disjunctive set has the form

$$S_4 = \bigcup_{i=1}^{m} \left\{ x| \sum_{j=1}^{n} a_{i,j}x_j \geq b_i,\ x \geq 0 \right\}.$$

As a matter of fact, these two representations are equivalent. Notice that the S_4 set can be nonconvex.

Disjunctive cuts

To formulate the principles outlined below, it is useful to consider a specific example.

Example 1. Let there be two inequalities for the non-negative variables x_1, x_2, x_3:

$$2x_1 + (-3)x_2 + (-7)x_3 \geq 15, \tag{1}$$

and

$$1x_1 + 2x_2 + (-5)x_3 \geq 28. \tag{2}$$

One of them is known to be true, but it is not known which one it is. The question is whether there is at least one inequality which is necessarily true.

We assert that such an inequality exists, and in this case it is a third inequality, for example, of the form

$$2x_1 + 2x_2 + (-5)x_3 \geq 15, \tag{3}$$

which is a relaxation of both inequalities (1) and (2), since

$$2 = \max\{2, 1\},\ \ 2 = \max\{-3, 2\},$$
$$-5 = \max\{-7, -5\}H15 = \min\{15, 28\}.$$

Thus, inequality (3) holds true in the event that at least one of inequalities (1) and (2) is true.

The DC metaprinciple generalizing the idea of obtaining the last inequality (3) from inequalities (1) and (2) can be formulated in the following way.

The DC metaprinciple. Let us assume that at least one of the inequalities

$$\sum_{j=1}^{r} b_{i,j}x_j \geq b_{i,0},\ \ i = 1, 2, \ldots, p,\ \ p \geq 2 \tag{4}$$

holds, and that the variables x_1, x_2,..., x_r are non-negative.

Then the inequality

$$\sum_{j=1}^{r} \left(\max_{i=1}^{p} b_{i,j} \right) x_j \geq \min_{i=1}^{p} b_{i,0} \tag{5}$$

and all its relaxations are valid cuts for the set

$$S_4 = \left\{ x| \bigvee_{i=1}^{p} \left(\sum_{j=1}^{r} b_{i,j}x_j \geq b_{i,0}, x \geq 0 \right) = \text{true} \right\}.$$

Since the DC principle, which is key for our study, follows from the LP principle and the DC metaprinciple [3,5], let us first formulate the LP principle.

The LP principle. If λ is an arbitrary non-negative m-vector, then the inequality

$$(\lambda A)x \geq \lambda b$$

and all its relaxations are valid inequalities for a non-empty set

$$S_1 = \{x|Ax \geq b, x \geq 0, x \in \mathrm{R}^r\}.$$

So let us now formulate the disjunctive cuts principle.

The DC principle [2,3,5]. *Let us assume that at least one of the systems of linear inequalities*

$$\left(A^h x \geq b^h, x \geq 0\right), h \in H, H \neq \emptyset \tag{6}$$

holds (A^h is a $m_h \times r$ matrix, b^h is a m_h column vector).

Then for any m_h row vectors λ^h with non-negative multipliers the inequality

$$\left(\sup_{h\in H} \lambda^h A^h \right) x \geq \inf_{h\in H} \lambda^h b^h \tag{7}$$

and all its relaxations are valid cuts for a disjunctive set

$$S_4 = \left\{ x| \bigvee_{h\in H} \left(A^h x \geq b^h, x \geq 0\right) = \text{true} \right\}.$$

Here and elsewhere we shall use the concept of a supremum of vectors. In inequality (7), $\sup_{h\in H} \lambda^h A^h$ denotes the v vector whose jth component is

$$v_j =^{\sup}_{h\in H} v_j^h (j = 1, 2, \ldots, r),\ \text{a}\ v^h = \lambda^h A^h,$$