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The exact analytical solution of the problem on the average number of spikes of the narrowband Gaussian stochastic process☆

Ivan D. Lobanov*, Alexander V. Denisov

Peter the Great St. Petersburg Polytechnic University, 29 Politekhnicheskaya St., St. Petersburg 195251, Russian Federation

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Abstract

In this article, the problem of the number of spikes (level crossings) of the stationary narrowband Gaussian process has been considered. The process was specified by an exponentially-cosine autocorrelation function. The problem had been solved earlier by Rice in terms of the joint probabilities' density of the process and its derivative with respect to time, but in our article we obtained the solution using the functional of probabilities' density (the functional was obtained by Amiantov), as well as an expansion of the canonical stochastic process. In this article, the optimal canonical expansion of a narrowband stochastic process based on the work of Filimonov and Denisov was also considered to solve the problem. The application of all these resources allowed obtaining an exact analytical solution of the problem on spikes of stationary narrowband Gaussian process. The obtained formulae could be used to solve, for example, some problems about the residual resource of some radiotechnical products, about the breaking sea waves and others.

Keywords: Emission of stochastic process; Density functional of probability; Canonical expansion.

The problem of the average number of spikes (level crossings) of a stationary stochastic process was first solved by Rice [1] in terms of joint probability density $W(\xi(t), \dot{\xi}(t))$ of the process $\xi(t)$ and its derivative with respect to time $\dot{\xi}(t)$. This solution was included in many textbooks and review papers on statistical radio engineering and radio physics, for example, in Refs. [2–4].

This problem still has not lost its relevance. The formula for finding the number of level crossings can be used for solving the following applied problems [5–7]:

- about the residual operating time of some radio hardware (e.g., expensive massive transformers);
- about the breaking marine (oceanic) gravitational waves (if the solution of these problems is based on spectral methods).

However, in obtaining the formula for the number of level crossings on a sufficiently short interval Δt in Ref. [3, p. 456], the differential $d\dot{\xi}$ was used instead of the increment of a random process derivative $\Delta\dot{\xi}$, but the limiting transfer operation $\Delta t \to 0$ was not

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* Corresponding author.
E-mail addresses: lobanov.111@yandex.ru (I.D. Lobanov), A.V.Denisov@inbox.ru (A.V. Denisov).

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performed, which in our opinion was not quite correct. As a result of this, Rice's formula became approximate.

In solving such problems, the researcher normally uses a power spectral density graph and (or) that of an autocorrelation function.

In view of this, it is of interest to obtain the exact analytical solution of this problem using the probability density functional [8] and the canonical decomposition of a random process [9,10].

Finding the probability density functional $F(x(t))$ which shows the relative probability of a trajectory appearing is, for an arbitrary autocorrelation function, a fairly complex mathematical problem related to solving the Fredholm integral equation. However, Ref. [8] obtained an exact expression for the probability density functional. This expression will be used as a basic one in this work and will be presented below.

Let us examine the trajectories of a stationary narrowband Gaussian stochastic process $x(t)$, given by the power spectral density (1) and its corresponding autocorrelation function (2) on the $[-T/2, T/2]$ interval:

$$\Phi(f) = \frac{2a\sigma^2}{(2\pi)^4(f-f_0)^2 + (2\pi)^2 f^2}, \tag{1}$$

$$K(\tau) = \sigma^2 \exp(-a|\tau|)\left(\cos(\omega_1\tau) + \frac{a}{\omega_1}\sin(\omega_1|\tau|)\right), \tag{2}$$

where
$\omega_0 = 2\pi f_0$, $\omega_0^2 = \omega_1^2 + a^2$,

Let us introduce the notation

$$H_0 = \exp\left[-\frac{1}{4\sigma^2 a^2\omega_0^2}\left(\left[a\omega_0^2 x_1^2 + a x_1^2\right] + \left[a\omega_0^2 x_2^2 + a x_2^2\right]\right)\right],$$

where x_1 and x_2 are, respectively, the initial and the final values of the trajectory of the random process $x(t)$ on some interval $[-T/2, T/2]$.

Taking into account the notation introduced (for this interval), the probability density functional of the process (1), (2) has the following form [8]:

$$F(x(t)) = hH_0 \exp\left[-\frac{1}{4\sigma^2 a^2\omega_0^2}\left(\int_{-\frac{T}{2}}^{\frac{T}{2}} x''^2(t)dt + a^2\int_{-\frac{T}{2}}^{\frac{T}{2}} x'^2(t)dt + \omega_0^4\int_{-\frac{T}{2}}^{\frac{T}{2}} x^2(t)dt + 2\omega_0^2\int_{-\frac{T}{2}}^{\frac{T}{2}} x''(t)dt\right)\right], \tag{3}$$

where $x'(t)$ and $x''(t)$ are the first and the second derivatives of $x(t)$; the interval $[-T/2, T/2]$ is arbitrary; h is a parameter depending on the partition rank of an n-dimensional function of the distribution of the stochastic process under consideration at $n \to \infty$, identical for different implementations of $x(t)$ [8].

From now on we shall assume that the T value is equal to the first-harmonic period T_0 of the canonical expansion of a stochastic process in terms of a trigonometric basis.

Let us write the general form of the canonical expansion of the narrowband process $x(t)$ with a zero mathematical expectation in this basis:

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos\left(\frac{2\pi k}{T_0}t + \varphi_k\right), \tag{4}$$

the coefficients A_0 and A_k in this basis are Gaussian random values with a zero mathematical expectation and variances σ_k^2, uncorrelated with each other; φ_k are random values uniformly distributed in the $[0; 2\pi]$ interval.

The variances σ_k^2 are found from the well-known autocorrelation function [9]:

$$\sigma_0^2 = 2\int_0^{\frac{T}{2}} \sigma^2 \exp(-a|\tau|) \times \left(\cos(\omega_1\tau) + \frac{a}{\omega_1}\sin(\omega_1|\tau|)\right)d\tau, \tag{5}$$

$$\sigma_k^2 = 2\int_0^{\frac{T}{2}} \sigma^2 \exp(-a|\tau|) \times \left(\cos(\omega_1\tau) + \frac{a}{\omega_1}\sin(\omega_1|\tau|)\right)\cos\left(\frac{2\pi k}{T}\tau\right)d\tau. \tag{6}$$

Excluding from expansion (4) the terms of the series for which $\sigma_k^2 \ll 1$, the expansion $x(t)$ can be approximately represented as a finite sum:

$$x(t) = \sum_{k=M}^{N} A_k \cos\left(\frac{2\pi k}{T_0}t + \varphi_k\right), \tag{7}$$

with

$$\sum_{k=M}^{N} \sigma_k^2 \approx \sigma^2.$$

It is known from Ref. [9] that the following conditions should be fulfilled to optimally choose the T_0 period of the canonical expansion: