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Rough estimates of the Blasius constant

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Abstract

Integral properties of homogeneous solutions of the Crocco boundary problem and splitting (flat) expansion have been used for an approximate estimate of the Blasius constant. The derivative *d*(fi)/*dh* was proved to have a logarithmic singularity at the point $h = 1$, therefore the second one tends to minus zero, and the function in itself tends to plus zero, because *h* tends to unity minus zero, so the splitting series is not slower to diverge as compared with the harmonic one. The existence of an integral invariant was proved for a uniform solution of the Crocco boundary problem, the solution exhibiting the squared norm of the solution derivative. The condition for the distribution minimum was established to be satisfied along the real uniform solutions of the Crocco boundary problem.

Keywords: Crocco boundary problem; Splitting decomposition; Distribution density; Homeomorphism; Boundary problem invariant.

1. Introduction and problem statement

Estimating the Blasius constant is crucial for determining the densities of mass, impulse and energy flow at the solid–liquid interface. The first value of the constant was determined by G. Blasius in a hydrodynamic problem of a boundary layer on a plate. Blasius used a power series connection, or analytic continuation, near the wall and the asymptotic expansion in the outer (jet) part of the flow. This is the way the Blasius solution is described in hydrodynamics courses by N.E. Kochin, L. Prandtl, and others. The history of the problem is outlined in [1,2].

We can assume that the exact value of the *a* constant is known and reliably calculated to thirty positions, and, as the author of [1] states, ‘The Blasius method is fully exonerated’.

Let us have

$$D(\varphi) = (h : 0 < h < 1)), \quad \varphi \in C^2(0, 1),$$
$$J(\varphi) = (\varphi : 0 < \varphi < a < \infty)$$

and $a := \phi(0)$ is the Blasius constant.

The diffeomorphism $\phi(h)$, $\phi \in C^{(2)}(0,1)$ is obtained as a solution of a two-point boundary problem:

$$2\varphi d^2\varphi/dh^2 + f(h) = 0, \quad (d\varphi/dh)_{h=0} = \varphi(1) = 0. \tag{1}$$

A two-point boundary problem can be reduced to a Cauchy problem with the following initial conditions:

$$\varphi(0) - a = (d\varphi/dh)_{h=0} = 0, \tag{2}$$

in this case the condition $\phi(1) = 0$ must be fulfilled.

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As Varin has suggested in [1] it is more convenient to lay down normalized initial conditions at the point $h = 1$:

$$\varphi(0) - 1 = (d\varphi/dh)_{h=0} = 0, \tag{2a}$$

where $0 < h < r$, $r = a^{-2/3}$, $\phi(r) = 0$.

The main problem is getting a precise estimate of a (the Blasius constant). This constant appears explicitly only in the reduction of the boundary problem (1) to a Cauchy problem; the problem (1) is convenient because all of the boundary conditions are trivial. If we normalize the constant a to 1, it is necessary to find the value r that is a root of $\phi(h)$. Moreover, in [2] the author proves that $\phi(h)$ is a distribution that is analytic in the interval $0 < h < r$. The point $h = r$ (or $h = 1$ in a non-normalized definition) is singular for the $d\phi/dh$ derivative [2].

In other words, the following result is given in [2]: let $\phi(h)$ be the solution of the boundary problem (1). Then $h = 1$ is a singular point,

$$\varphi(1) = 0, \quad -d\varphi/dh \to \infty, \quad h \to 1 - 0,$$

i.e. $r = 1$ is a natural convergence radius depending on a:

$$\forall\varepsilon > 0, \quad \exists\eta(\varepsilon) > 0,$$
$$\eta \underset{\varepsilon\to+0}{\to} +0 \Rightarrow 1 - \eta < r(a+\varepsilon) < r(a-\varepsilon) < 1 + \eta.$$

The method of the so-called ‘flat expansions’ is used to calculate the constant a [3]. The method is convenient as it allows limiting the calculations to a small number of left-side series members.

This approximation is ensured by the uniform convergence of the flat series. The uniform convergence is proved in [3,4]. Similar function series (splitting expansions) are described in [5]; the concept of using the expansion in the form that it is done in this paper comes from Schwartz and Huet [6].

The goal of the current paper is to obtain the approximate estimates for the a constant. This problem is solved by using the direct method based on calculating the distribution norms and distribution densities. Hereinafter we shall assume that the estimates are unfit if the calculation error of a is not less than 1 %.

2. The main properties of a boundary problem solution (1)

As a first step of our study, let us examine the solution properties of an equation

$$2\frac{\varphi d^2\varphi}{dh^2} + h = 0, \tag{3}$$

with the following set of limiting conditions:

$$(d\varphi/dh)_{h=0} = \varphi(1) = 0,$$
$$\varphi(0) - a = \varphi(1) = 0,$$
$$(d\varphi/dh)_{h=0} = \varphi(0) - a = 0. \tag{4}$$

Renormalization to a unit-length interval is not expedient in this case, as $r = a^{-2/3}$ (the a and the r constants are connected).

Lemma 1. *A formal first integral of Eq.* (3), *such that*

$$(d\varphi/dh)_{h=0} = 0,$$

is of the following form:

$$\psi^2 := (d\varphi/dh)^2 = \int_0^\omega h(\tau)\,d\tau, \ \omega := \ln(a/\varphi) \in (0, \infty). \tag{5}$$

The proof of Lemma 1 is obvious.

It follows from Lemma 1 that

$$\exp(-\omega)\frac{d\omega}{dh} = -\sqrt{\int_0^\omega h(\tau)\,d\tau},$$

and it is fairly easy to obtain a non-linear integral equation to determine $h(\omega)$.

The distribution $h(\omega) \geq 0$ is monotonous, i.e.

$$\frac{dh}{d\omega} > 0, \quad h(0) = h(\infty) - 1 = 0,$$

therefore, by virtue of the second law of the mean (see [7]):

$$\exists\omega^* \in (0, \omega) \Rightarrow \int_0^\omega h(\tau)\,d\tau = h(\omega)(\omega - \omega^*).$$

Let us introduce a notation

$$\sigma := 1 - \frac{\omega^*}{\omega} < 1,$$

or, in another form,

$$\sigma := \frac{\int_0^\omega h(\tau)\,d\tau}{\omega h(\omega)}. \tag{6}$$

Lemma 2. *The Eq. (3) is equivalent to a canonical system with the following Hamiltonian:*

$$H(\omega, \psi, h) = 1/2\left(\psi^2 - \omega h(\omega)\right) < 0.$$

Proof. Indeed,

$$\psi^2 - \omega h(\omega) = (\omega^* - \omega)h(\omega) = -\sigma\omega h(\omega) < 0,$$

□

Corollary to Lemma 2. An actual path (characteristic) of Eq. (3) satisfies the condition of the $Z(1,\phi,\psi)$ distribution minimum:

$$Z(1, \varphi, \psi) = \int_0^1 \left(\psi^2 + h\ln(a/\varphi)\right) dh \to \inf > 0.$$