Available online at www.sciencedirect.com
ScienceDirect
Journal of Magnetism and Magnetic Materials 306 (2006) 16–23
Journal of magnetism and magnetic materials
www.elsevier.com/locate/jmmm

Quantum theory of spin waves in finite samples

D.L. Mills*

Department of Physics and Astronomy, University of California, Irvine, CA 92697, USA

Received 29 November 2005; received in revised form 30 January 2006
Available online 5 April 2006

Abstract

We present the formalism for the quantum theory of spin waves in finite samples of arbitrary shape. The sample shape is assumed such that the magnetization per unit volume and the internal demagnetizing field are constant in direction and magnitude everywhere, though this restriction may be lifted. We proceed within the framework of continuum theory, with both dipolar interactions and exchange interactions between the spins included. We derive a prescription for normalizing the spin wave eigenfunctions, and also provide a representation of the operators associated with the transverse components of the magnetization density. Completeness relations, which form the basis of expansion of arbitrary functions in terms of spin wave eigenfunctions, are derived as well. The theory may be employed to describe the interaction of spin wave quanta with external probes and other phenomena where the quantum nature of spin waves enters. We use the formalism to obtain an expression for the spatial and temperature dependence of the magnetization within a ferromagnetic nanosphere, at low temperatures where spin wave theory is applicable. We explore this issue with an explicit calculation.

Keywords: Spin waves; Normalized eigenfunctions; Thermal fluctuations

1. Introduction

In an ordered magnet, of course it has been known for decades that at sufficiently low temperatures, spin waves are the elementary excitations responsible for the contribution of the magnetic degrees of freedom to the thermodynamic properties of the material, and they also control its response to diverse external probes. In ferromagnets, under conditions where both dipolar and exchange interactions must be considered, perhaps the most complete and clearest discussion of the quantum theory of these excitations may be found in the classic paper of Holstein and Primakoff [1]. With their theoretical structure in hand, one may describe physical process in which spin wave quanta and emitted or absorbed, along with the spin wave contribution to the thermodynamic properties of the ferromagnet. The formalism makes use of the plane wave description of spin waves, appropriate to the infinitely extended bulk crystal.

In the current era, where nanoscale magnetic structures are the focus of many studies, the spin wave excitations of small samples of diverse shape have been explored by many authors, for reasons similar to those which motivated the authors of Ref. [1]. On nanometer length scales, it is necessary to include the influence of exchange interactions between the spins, as well as the dipolar interactions between them. The present author and his collaborators have developed analytic descriptions of the dipole/exchange modes of cylindrical wires [2] and spheres [3], and also the magnetostatic modes of ferromagnetic ribbons of arbitrary cross section [4]. Various others have used numerical methods to obtain the frequencies of the dipole/exchange spin waves of finite structures, along with their associated eigenfunctions [5–7].

The theoretical studies cited in the previous paragraph solve the linear eigenvalue problem associated with the description of spin wave excitations. While the eigenvalues and eigenfunctions so generated provide very useful insight into the response characteristics of the sample, the theory is incomplete. One desires a complete quantum theory of the spin waves, expressed in terms of the relevant boson

*Tel.: +714 824 5148; fax: +714 824 2174.
E-mail address: dlmills@uci.edu.

0304-8853/$ - see front matter © 2006 Published by Elsevier B.V.
doi:10.1016/j.jmmm.2006.02.267

creation and annihilation operators associated with the various modes. Without such a theory, one cannot use the information generated in these studies to obtain a description of physical processes in which spin wave quanta are annihilated or created. One cannot obtain a description of thermodynamic properties of the material as well, at low temperatures where application of spin wave theory is appropriate. In the latter area, there are issues of interest. For example, in the ground state, the magnetization will be constant in magnitude and direction everywhere, in samples of suitable shape. However, at finite temperatures, the amplitude of the thermal fluctuations will vary from point to point in the sample, with the consequence that the magnitude of the magnetization will vary as one scans about the sample. To obtain a description of such an effect, one requires a quantum theory of spin waves, which includes a normalization condition on the eigenfunctions not present in the analyses examined in papers such as those in Refs. [2–7].

In this paper, we present a quantum theory of spin waves for samples of arbitrary shape, within the framework of the continuum theory of spin motions with both dipolar and exchange interactions between the spins included. We do confine our attention to samples within which the magnetization per unit volume is constant in both magnitude and direction everywhere, in the ground state at the absolute zero of temperature. It is our view that it should be possible to extend this discussion to the more general case where the magnetization varies in direction from point to point, but we leave this to future studies.

We should comment on the motivation for this study. We were presented with the need for such a theoretical structure during the course of the development of the theory of Brillouin light scattering from the dipole exchange modes of ferromagnetic nanospheres. This investigation will be described elsewhere. We were attracted by intriguing, very brief remarks in Appendix II of Walker's classic paper [8] on the magnetostatic modes of ferromagnetic ellipsoids. In this short Appendix, two striking and unusual orthogonality relations are displayed. Its final sentence remarks that with these two relations in hand, one may diagonalize the Hamiltonian of the system, though no details are given. We show here this is in fact that case, also when exchange is included between the spins. This provides us with the starting point for the development of the quantum theory of spin waves described below. We are also led to new completeness relations and orthogonality relations not noted by Walker. When all this is assembled, we have in hand a formal structure which may be used to describe spin wave quanta and their interaction with external probes in samples of diverse shape, along with a means to expand any arbitrary function in terms of their eigenfunctions. The author views the present discussion as a exploration of the consequences of the final sentence of a most remarkable paper, studied by many for decades for reasons rather different than ours.

2. The formalism

As noted in Section 1, we confine our attention to samples in which the magnetization per unit volume, $\vec{M}_{\rm S}$, is constant in magnitude and direction throughout the sample, in the ferromagnetic ground state at the absolute zero of temperature. We also assume that the internal field $\vec{H}_{\rm I}$ is constant in magnitude and direction everywhere, and parallel to the magnetization. The z-axis of the coordinate system is aligned parallel to this common direction. The spin motions may be described by the magnetization densities $m_x(\vec{r},t)$ and $m_y(\vec{r},t)$. These are viewed for the moment as classical variables, and later in the discussion we shall make the transition to regarding them as quantum mechanical operators. As the spins precess, they generate a time-dependent magnetic dipole field $\vec{h}_{\rm d}(\vec{r},t)$. In the magnetostatic limit, this field may be expressed as the gradient of a magnetic potential $\Phi_{\rm M}(\vec{r},t)$, $\vec{h}_{\rm d}(\vec{r},t)=-\vec{\nabla}\Phi_{\rm M}(\vec{r},t)$.

The equations of motion of the magnetization are well known forms. In the spin wave limit where the equations are linearized:

$$\frac{\partial m_x}{\partial t}=-\gamma\left(H_{\rm I}-D\nabla^2\right)m_y-\gamma M_{\rm S}\frac{\partial \Phi_{\rm M}}{\partial y}, \tag{1a}$$

$$\frac{\partial m_y}{\partial t}=\gamma\left(H_{\rm I}-D\nabla^2\right)m_x+\gamma M_{\rm S}\frac{\partial \Phi_{\rm M}}{\partial x}, \tag{1b}$$

while the magnetic potential is found from

$$-\nabla^2\Phi_{\rm M}+4\pi\left(\frac{\partial m_x}{\partial x}+\frac{\partial m_y}{\partial y}\right)=0. \tag{2}$$

The magnetic potential obeys Laplace's equation outside the sample, and we have the boundary conditions that tangential components of $\vec{h}_{\rm d}$ are conserved, along with the normal component of $\vec{b}_{\rm d}=\vec{h}_{\rm d}+4\pi\vec{m}$. We also append the free spin boundary conditions on the magnetization at the sample surface. So we require that

$$\left.\frac{\partial m_x}{\partial n}\right|_S=\left.\frac{\partial m_y}{\partial n}\right|_S=0. \tag{3}$$

It should be remarked that the constructions described below all go through unchanged in structure if surface anisotropy is incorporated into the boundary condition in Eq. (3), and in the energy functional described below. We omit this complication in the presentation here, in the interest of simplicity. Finally, we use a convention where the gyromagnetic ratio γ is a positive number.

The spin wave eigenfrequencies $\{\Omega_\lambda\}$ are found by seeking solutions of Eqs. (1) and (2) of the form $m_{x,y}(\vec{r},t)=m^\lambda_{x,y}(\vec{r})\exp(-{\rm i}\Omega_\lambda t)$ and similarly for the magnetic potentials. Thus, the eigenvalue equations one examines are given by

$$ {\rm i}\Omega_\lambda m^\lambda_x(\vec{r})=\gamma\left(H_{\rm I}-D\nabla^2\right)m^\lambda_y(\vec{r})+\gamma M_{\rm S}\frac{\partial \Phi^\lambda_{\rm M}}{\partial y}, \tag{4a}$$