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Green's functions of the scalar model of electromagnetic fields in sinusoidal superlattices

V.A. Ignatchenko *, D.S. Tsikalov

L.V. Kirensky Institute of Physics, Siberian Branch of RAS, 660036 Krasnoyarsk, Russia

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ABSTRACT

Problems of obtaining Green's function and using it for studying the structure of scalar electromagnetic fields in a sinusoidal superlattice are considered. An analytical solution of equation in the **k**-space for Green's function is found. Green's function in the **r**-space is obtained by both the numerical and the approximate analytical Fourier transformation of that solution. It is shown, that from the experimental study of Green's function in the **k**-space the position of the plane radiation source relative to the extremes of the dielectric permittivity $\varepsilon(z)$ can be determined. The relief map of Green's function in the **r**-space shows that the structure of the field takes the form of chains of islets in the plane ωz, the number of which increases with increasing the distance from a radiation source. This effect leads to different frequency dependences of Green's function at different distances from the radiation source and can be used to measure the distance to the internal source. The real component of Green's function and its spatial decay in the forbidden zones in the near field is investigated. The local density of states, depending on the position of the source in the superlattice, is calculated.

1. Introduction

Photonic crystals, which are artificially created media with periodic physical parameters, has recently been widely studied (see, e.g., Refs. [1–6]). Knowledge of analytical expressions for Green's function of waves propagating in such media, is necessary when considering a number of problems, both theory and experiment. Green's function in the coordinate space $G(\omega, \mathbf{r}, \mathbf{r}_0)$ is used as in dealing with problems related to the structure of wave fields in periodic media and computing the most important characteristics such as the local density of states (LDOS) [7–10]. Green's function in the wave vector space $G(\omega, \mathbf{k}, \mathbf{r}_0)$ is needed in the study of various aspects of theory, for example, the theory of wave scattering by inhomogeneities [11,12]. The wave equation for Green's function in one-dimensional superlattice, periodic along the *z*-axis, is reduced to a one-dimensional equation by the Fourier transformation in the $\xi = x - x_0$ and $\zeta = y - y_0$ coordinates. This work is devoted to finding and investigating Green's functions for the scalar model of electromagnetic waves in one-dimensional superlattice with a sinusoidal profile of modulation of the dielectric permittivity $\varepsilon(z)$. Real photonic crystals typically have a modulation profile $\varepsilon(z)$, nearly rectangular. A number of physical phenomena occurring in the propagation of waves in periodic media, are very sensitive to the shape of the profile of $\varepsilon(z)$. However, a number of phenomena, including the fundamental nature, are qualitatively similar for all periodic profiles of $\varepsilon(z)$. The sinusoidal modulation of $\varepsilon(z)$ is the most suitable for the analytical study of such phenomena. A homogeneous equation for a sinusoidal superlattice (Mathieu equation) is well studied (see, e.g., Ref. [13]). Green's function of waves in this superlattice is much less studied. In Refs. [14–18], approximate expressions for Green's function in the coordinate space $G(\omega, \mathbf{r}, \mathbf{r}_0)$ have been found and studied.

The objectives of this paper are: (i) obtaining an analytical representation of Green's function in **k**-space $G(\omega, \mathbf{k}, \mathbf{r}_0)$ for scalar waves in the one-dimensional sinusoidal superlattice; (ii) the numerical and approximate analytical representation of this function in the **r**-space $G(\omega, \mathbf{r}, \mathbf{r}_0)$; (iii) investigation of the structure of the scalar fields of the plane radiation source using both forms of Green's function, $G(\omega, \mathbf{k}, \mathbf{r}_0)$ and $G(\omega, \mathbf{r}, \mathbf{r}_0)$.

2. Solution of Green's function equation

Green's function of scalar model of electromagnetic waves in a sinusoidal superlattice satisfies the equation

$$\nabla^2 G(\mathbf{r}, \mathbf{r}_0) + \left[\nu + 2\eta\cos(qz + \psi)\right]G(\mathbf{r}, \mathbf{r}_0) = -\delta(\mathbf{r} - \mathbf{r}_0), \tag{1}$$

where $\nu = \varepsilon(\omega/c)^2$, $2\eta = \Delta\varepsilon(\omega/c)^2$, $q = 2\pi/l$; ω and c are the

* Corresponding author.
E-mail address: vignatch@iph.krasn.ru (V.A. Ignatchenko).

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frequency and the speed of light in vacuum, respectively; ε and $\Delta\varepsilon$, respectively, are the mean value and the amplitude of modulation of a dielectric permittivity, l and ψ are the spatial period and the phase of the superlattice, respectively. Here and below, we do not indicate explicitly the dependence of Green's functions on the frequency if it does not lead to misunderstandings. In addition to Eq. (1), Green's function must satisfy the standard conditions of radiation. Equations for scalar models of elastic and spin waves differ renaming parameters.

Since the medium is periodically inhomogeneous along the superlattice z-axis, Green's function depends not only on the difference $z - z_0$, and the z-coordinate directly. In the xy plane, Green's function depends only on the difference of the corresponding coordinates, that allows us to carry out the two-dimensional Fourier transformation in the transverse coordinates $\xi = x - x_0$ and $\zeta = y - y_0$:

$$G(\mathbf{r}, \mathbf{r}_0) = \int G(\mathbf{k}_\perp, z, z_0) \exp\left[i(\xi k_\xi + \zeta k_\zeta) \right] dk_\xi\, dk_\zeta. \quad (2)$$

The result is a one-dimensional equation in the form

$$\frac{d^2}{dz^2} G(\mathbf{k}_\perp, z, z_0) + \left[\varkappa + 2\eta \cos(qz + \psi) \right] G(\mathbf{k}_\perp, z, z_0) = -\frac{1}{(2\pi)^2} \delta(z - z_0), \quad (3)$$

where $\varkappa = \nu - k_\perp^2$ and $\mathbf{k}_\perp = \mathbf{i}k_\xi + \mathbf{j}k_\zeta$ is a two-dimensional wave vector. Eq. (3) was used in Refs. [16,18], where the approximate expressions for Green's function in some limiting cases had been obtained. In those studies, Eq. (3) was investigated in the coordinate z-space. In that case, the solution of Eq. (3) was expressed in terms of independent solutions of the corresponding homogeneous equation [13]. These cumbersome expressions are not always convenient both in analytical and numerical calculations.

In this paper, we develop another method for studying the solution of Eq. (3), which we briefly have described previously [19]. First, we find the analytical solution of Eq. (3) in k_z-space, and then examine it numerically and analytically in $\mathbf{r}$-space. Applying the Fourier transformation to Eq. (3)

$$G(z) = \int G(k_z) \exp(ik_z z)\, dk_z, \quad G(k_z) = \frac{1}{2\pi} \int G(z) \exp(-ik_z z)\, d_z, \quad (4)$$

we obtain

$$\left[\varkappa - k_z^2 \right] G(k_z) + \eta \left[e^{i\psi} G(k_z - q) + e^{-i\psi} G(k_z + q) \right] = -\frac{1}{(2\pi)^3} \exp\left[-ik_z z_0 \right]. \quad (5)$$

To find the solution of Eq. (5), we use the methods of analysis of systems of matrix equations [20,21]. Doing the corresponding operations (see Appendix A), we obtain Green's function in $\mathbf{k}$-space in a compact expression containing the ascending and ordinary continued fractions

$$G(\mathbf{k}_\perp, k_z, z_0) = -\frac{\exp(-ik_z z_0)}{(2\pi)^3} \frac{1 + P_1^+ + P_1^-}{L_0}. \quad (6)$$

Here $P_1^\pm$ are ascending continued fractions, determined by the recursive formula

$$P_n^\pm = -\eta \exp(\pm i\psi) \frac{\exp(\pm inqz_0) + P_{n+1}^\pm}{L_n^\mp}, \quad (7)$$

and L_0 and $L_n^\pm$ are ordinary continued fractions defined by the formulas

$$L_0 = \varkappa - k_z^2 - \eta^2 \left[\frac{1}{L_1^+} + \frac{1}{L_1^-} \right], \quad L_n^\pm = \varkappa - (k_z \pm nq)^2 - \frac{\eta^2}{L_{n+1}^\pm}. \quad (8)$$

Continued fractions in Eq. (6) have fast convergence, so that expression is useful in the study of Green's function in $\mathbf{k}$-space and in the $\mathbf{r}$-space. In some cases it is convenient to use also the expansion of Green's function in a Fourier series, which has the form

$$G(\mathbf{k}_\perp, k_z, z_0) = \sum_{n=-\infty}^{\infty} g_{\mathbf{k}}^n \exp\left[-i(k_z - nq) z_0 \right]. \quad (9)$$

Here the factors $g_{\mathbf{k}}^n$ are determined by the expression

$$g_{\mathbf{k}}^{\pm n} = -\frac{1}{(2\pi)^3} \left[-\eta \exp(\pm i\psi) \right]^n \Big/ L_n^\mp \Big/ L_{n-1}^\mp \Big/ \cdots \Big/ L_0, \quad n = 0, 1, 2, \ldots, \quad (10)$$

where continued fractions designated by slash characters.

3. Field structures of a plane radiation source

In what follows, all graphs of Green's functions correspond to the plane radiation source located in the plane xy. Green's function of the source in the z-space defined by the equation

$$G(z, z_0) = \iint G(\mathbf{r}, \mathbf{r}_0)\, dx_0\, dy_0. \quad (11)$$

Substituting Eq. (2) into Eq. (11) and integrating over x_0 and y_0, we obtain

$$G(z, z_0) = (2\pi)^2 \iint G(\mathbf{k}_\perp, z, z_0) \exp\left[i(xk_\xi + yk_\zeta) \right] \delta(k_\xi) \delta(k_\zeta)\, dk_\xi\, dk_\zeta = (2\pi)^2 G(\mathbf{k}_\perp, z, z_0)|_{\mathbf{k}_\perp = 0}. \quad (12)$$

It follows that Green's functions of a plane source $G(k_z, z_0)$ and $G(z, z_0)$ related to the general expression for the spectral form of Green's function, Eq. (6), by the following relations:

$$G(k_z, z_0) = (2\pi)^2 G(\mathbf{k}_\perp, k_z, z_0)|_{\mathbf{k}_\perp = 0}, \quad (13)$$

$$G(z, z_0) = (2\pi)^2 \int G(\mathbf{k}_\perp, k_z, z_0)\Big|_{\mathbf{k}_\perp = 0} \exp(ik_z z)\, dk_z. \quad (14)$$

Examples of relief maps of the imaginary part of Green's function $G''(\omega, k_z)$ in k_z-space calculated by Eqs. (6) and (13) are shown in Fig. 1 for the three phase values: $\psi = 0$ (a), $\pi/2$ (b), and π (c). For the expressiveness of maps, the dimensionless factor ω/ω_r is added to the normalization of Green's functions, where ω_r corresponds to the middle frequency of the first forbidden Brillouin zone of the superlattice. Without such a leveling factor, the amplitudes of the relief in the high-Brillouin zones are too small. In the calculations, it is assumed that the source coordinate $z_0 = 0$ and the location of the source relative to the superlattice is governed by the spatial phase of the superlattice ψ. Phase $\psi = 0$ corresponds to the source position in one of the maxima of the function $\cos qz$, i.e., in a center of the layer with a large value of ε, the phase $\psi = \pi/2$ corresponds to the source position at the boundary between the layers, and the phase $\psi = \pi$ corresponds to the source position in a center of the layer with a lower value of ε (see the image of a superlattice in the bottom of Fig. 2). In the color version of Fig. 1, available online, one can see that the phase change leads to radical restructuring the relief of the function $G''(\omega, k_z)$. At the sites of some positive for $\psi = 0$ peaks of this function, the negative peaks occur at $\psi = \pi$, the character of the sequence of peak sights changes as along the k_z coordinate, and along the ω coordinate. Especially peculiar pattern corresponds to the phase $\psi = \pi/2$, when instead of peaks the curves having both positive and negative components occur.

The spatial structure of the electromagnetic field in the superlattice along the z-axis describes by Green's function $G(\omega, z)$. This function is determined by the Fourier transformation in the