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Bilinear forms and soliton solutions for a fourth-order variable-coefficient nonlinear Schrödinger equation in an inhomogeneous Heisenberg ferromagnetic spin chain or an alpha helical protein

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ABSTRACT

In this paper, a fourth-order variable-coefficient nonlinear Schrödinger equation is studied, which might describe a one-dimensional continuum anisotropic Heisenberg ferromagnetic spin chain with the octuple–dipole interaction or an alpha helical protein with higher-order excitations and interactions under continuum approximation. With the aid of auxiliary function, we derive the bilinear forms and corresponding constraints on the variable coefficients. Via the symbolic computation, we obtain the Lax pair, infinitely many conservation laws, one-, two- and three-soliton solutions. We discuss the influence of the variable coefficients on the solitons. With different choices of the variable coefficients, we obtain the parabolic, cubic, and periodic solitons, respectively. We analyse the head-on and overtaking interactions between/among the two and three solitons. Interactions between a bound state and a single soliton are displayed with different choices of variable coefficients. We also derive the quasi-periodic formulae for the three cases of the bound states.

1. Introduction

Nonlinear Schrödinger (NLS)-type equations have been used for people to study the nonlinear sciences including fluids, plasmas, optics, condensed matter physics, particle physics, and biophysics [1–8]. For instance, to describe the dynamics of a one-dimensional continuum anisotropic Heisenberg ferromagnetic spin chain with the octuple–dipole interaction or the alpha helical protein with higher-order excitation and interaction under the continuum approximation, a fourth-order NLS equation [9–18],

$$iu_t + \frac{1}{2}u_{xx} + |u|^2u + \frac{\varepsilon^2}{24}(u_{xxxx} + 8|u|^2u_{xx} + 2u^2u^*_{xx} + 6u^*u_x^2 + 4u|u_x|^2 + 6|u|^4u) = 0, \quad (1)$$

has been used, where the variables x and t represent the scaled distance and the scaled time, respectively, u represents the coherent amplitude in Glauber's coherent-state representation for the Heisenberg ferromagnetic spin chain [9] or the probability amplitude of the excitation in the protein molecular chain [10–12], with ε being the expansion parameter [9] or lattice parameter [10], the subscripts and asterisk, respectively, denote the partial derivatives and complex conjugate. Eq. (1) has been derived from the system describing the isotropic Heisenberg ferromagnetic spin chain, with the Lax pair constructed [15], and from the alpha helical protein with higher-order excitation and interaction, with the bilinear forms and one-soliton solutions obtained [10–12]. Painlevé property of Eq. (1) has been investigated and perturbed soliton solutions have been obtained [16]. Nonlinear spin dynamics governed by Eq. (1) has indicated that the addition of discreteness effect, i.e., the fourth-order terms, can destroy the NLS integrability [9]. Existence of the periodic weak solutions of Eq. (1) has been proved [17]. Bright N-soliton solutions of Eq. (1) have been constructed, where the asymptotic analysis has been applied to the two-soliton solutions to illustrate that the interactions between the two solitons are elastic [13]. Dark multi-soliton solutions for the corresponding defocusing system of Eq. (1) have been

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derived, while head-on and overtaking interactions have been discussed graphically [14]. Via the Darboux transformation, one-, two- and three-soliton solutions of Eq. (1) have been obtained [18].

Methods to derive the soliton solutions of the NLS-type equations have been presented, such as the Darboux transformation [19–21], Bäcklund transformation [22], inverse scattering method [23] and Hirota method [24,25]. However, most of the aforementioned methods have only been related to the constant-coefficient NLS-type equations [26], while the studies on variable-coefficient NLS-type equations have been used to describe certain real situations in physical and engineering sciences [27]. For example, variable coefficients in Eq. (2) as below may arise due to the presence of additional molecules such as the drugs in specific sites of the alpha helical protein, distance between the neighbouring atoms may vary along the lattices, atomic wave functions may vary from site to site or there may be imperfections in the vicinity of the protein molecules [18]. In Ref. [28], an inhomogeneous generalized fourth-order NLS equation, i.e., Eq. (1) with an additional term $2u\int h_x|u|^2\,dx$ (h is related to x), has been considered for the inhomogeneous alpha helical protein or Heisenberg ferromagnetic spin chain, in which the integrability and N-soliton solutions have been derived and the propagation of the solitons have been discussed as well.

In this paper, with symbolic computation, we will investigate a fourth-order variable-coefficient NLS equation:

$$iu_t+\alpha(t)u_{xx}+\beta(t)u|u|^2+\gamma_1(t)u_{xxxx}+\gamma_2(t)|u|^2u_{xx}+\gamma_3(t)u^2u_{xx}^*+\gamma_4(t)u^*u_x^2+\gamma_5(t)|u_x|^2u_{xx}+\gamma_6(t)|u|^4u=0, \tag{2}$$

where $\alpha(t)$, $\beta(t)$, $\gamma_1(t)$, $\gamma_2(t)$, $\gamma_3(t)$, $\gamma_4(t)$, $\gamma_5(t)$ and $\gamma_6(t)$ are all the real functions of t, which might be used to describe an inhomogeneous one-dimensional continuum anisotropic Heisenberg ferromagnetic spin chain or alpha helical protein. In general, Eq. (2) is not integrable, but when we balance the linear dispersion terms with the nonlinear terms using the method in Ref. [29], we find that Constraints (7) can ensure Eq. (2) being integrable.

To our knowledge, Eq. (2) has not been studied in the existing literatures. Motivated by that, in our paper, under Constraints (7), bilinear forms, Lax pair, infinitely many conservation laws and one- two- and three-soliton solutions for Eq. (2) will be derived in Section 2. Evolution of the two and three solitons, including the head-on and overtaking interactions, will be discussed graphically in Section 3 with different choices of the variable coefficients. Stationary and bound-state solitons will be obtained in Section 4, where the quasi-periodic formulas for three cases will be derived and some corresponding results will be discussed. Our conclusions will be presented in Section 5.

2. Integrable constraints, bilinear forms, Lax pair, infinitely many conservation laws and soliton solutions for Eq. (2)

2.1. Integrable constraints and bilinear forms for Eq. (2)

In order to derive the bilinear forms for Eq. (2), we can introduce the dependent variable transformation $u(x,t)=g(x,t)/f(x,t)$ [24,25], where $g(x,t)$ is the complex function of x and t, $f(x,t)$ is a real one. Eq. (2) can be transformed into

$$\begin{aligned}&i\frac{D_tg\cdot f}{f^2}+\alpha(t)\left(\frac{D_x^2g\cdot f}{f^2}-\frac{g}{f}\frac{D_x^2f\cdot f}{f^2}\right)+\beta(t)\frac{g^2g^*}{f^2}\\&+\gamma_1(t)\left[\frac{D_x^4g\cdot f}{f^2}-6\frac{D_x^2g\cdot f}{f^2}\frac{D_x^2f\cdot f}{f^2}+6\frac{g}{f}\left(\frac{D_x^2f\cdot f}{f^2}\right)^2-\frac{g}{f}\frac{D_x^4f\cdot f}{f^2}\right]\\&+\gamma_2(t)\frac{gg^*}{f^2}\left(\frac{D_x^2g\cdot f}{f^2}-\frac{g}{f}\frac{D_x^2f\cdot f}{f^2}\right)\\&+\gamma_3(t)\frac{g^2}{f^2}\left(\frac{D_x^2g^*\cdot f}{f^2}-\frac{g^*}{f}\frac{D_x^2f\cdot f}{f^2}\right)+\gamma_4(t)\frac{g^*}{f}\left(\frac{D_xg\cdot f}{f^2}\right)^2\\&+\gamma_5(t)\frac{g}{f}\frac{D_xg\cdot f}{f^2}\frac{D_xg^*\cdot f}{f^2}+\gamma_6(t)\frac{g^3(g^*)^2}{f^5}=0,\end{aligned} \tag{3}$$

where D_x and D_t both are the bilinear derivative operators defined by [24,25]

$$D_x^mD_t^n\Phi(x,t)\cdot\Psi(x,t)\equiv\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^m\times\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^n\Phi(x,t)\,\Psi(x',t')|_{x'=x,t'=t}, \tag{4}$$

x' and t' are the formal variables, $\Phi(x,t)$ and $\Psi(x,t)$ are two differentiable functions of x and t, m and n are two non-negative integers. Setting $D_x^2f\cdot f=\kappa gg^*$ and using

$$\frac{D_t^4f\cdot f}{f^2}=\left(\frac{D_t^2f\cdot f}{f^2}\right)_{tt}+3\left(\frac{D_t^2f\cdot f}{f^2}\right)^2, \tag{5}$$

with κ being a positive constant, we can transform Expression (3) into two parts: one is the linear part

$$\wp=\frac{iD_tg\cdot f}{f^2}+\alpha(t)\frac{D_x^2g\cdot f}{f^2}+\gamma_1(t)\frac{D_x^4g\cdot f}{f^2}, \tag{6}$$

and the other is the nonlinear part. To balance the nonlinear terms and dispersive terms of Eq. (3), we hereby introduce the following integrable constraints which can be derived via the method in Ref. [29]:

$$\begin{aligned}&\gamma_2(t)=4\kappa\gamma_1(t),\quad \gamma_3(t)=\kappa\gamma_1(t),\quad \gamma_4(t)=3\kappa\gamma_1(t),\\&\gamma_5(t)=2\kappa\gamma_1(t),\quad \gamma_6(t)=\frac{3}{2}\kappa^2\gamma_1(t),\quad \beta(t)=\kappa\alpha(t),\end{aligned} \tag{7}$$

under which, the nonlinear part of Eq. (3) can be reduced to

$$\Re=-\frac{3\kappa\gamma_1(t)g^*}{2f^3}D_x^2g\cdot g. \tag{8}$$

From Constraints (7), we can find that there are only two independent variable coefficients for Eq. (2), so that it is possible for us to express the bilinear forms just including $\alpha(t)$, $\gamma_1(t)$ (and the independent constant just introduced, κ). From Eqs. (6) and (8), we can derive the bilinear forms for Eq. (2) under Constraints (7) as

$$\begin{aligned}&D_x^2f\cdot f=\frac{\beta(t)}{\alpha(t)}gg^*,\\&\left[iD_t+\alpha(t)D_x^2+\gamma_1(t)D_x^4\right]g\cdot f=\frac{3\beta(t)\gamma_1(t)}{2\alpha(t)}g^*s,\\&D_x^2g\cdot g=sf.\end{aligned} \tag{9}$$

In such situations as the drugs in specific sites of the alpha helical protein, pharmacological effects may differ as t changes [18], so that we need to consider the inhomogeneity in an alpha helical protein, where the terms related to $\alpha(t)$ represent the transfer of energy along the hydrogen bonding spine at the lowest order of the lattice parameter and the terms related to $\gamma(t)$ represent the higher-order excitations and interactions. Electromagnetic flux may change with t, which leads us to consider the inhomogeneous effects in a one-dimensional continuum anisotropic Heisenberg ferromagnetic spin chain, where the terms proportional to $\alpha(t)$ and $\gamma(t)$ represent the elementary spin