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One-phase flow in porous media with hysteresis

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ABSTRACT

This paper presents a numerical simulation of one phase flow through a porous medium showing a hysteretic relation between the capillary pressure and the saturation of the phase. The flow model used is based on mass conservation principle and Darcy's law. Boundary conditions of Neumann and Signorini type are imposed. The hysteretic relation between the capillary pressure and the saturation is described by a Preisach hysteresis operator. A numerical algorithm for the treatment of the arising system of equations is proposed. Results of numerical simulations are presented.

1. Introduction

The nature of flows in porous media is characterized by hysteretic effects observable on a macroscopic level. These effects have a significant influence on the behavior of the whole system, and therefore they have to be taken into account. The aim of this work is to study an evolution problem of filtration through porous media, accounting for hysteresis in the saturation versus pressure constitutive relation. The mass conservation principle and Darcy's law yield a nonlinear diffusion equation. This equation is then coupled with Neumann and Signorini boundary conditions imposed on different parts of the boundary. The relationship between the saturation and the pressure is modeled by a Preisach hysteresis operator.

The paper presents an existence result for a weak formulation in the framework of Sobolev spaces and proposes a numerical algorithm, which allows us to perform simulations of the coupled system.

2. Model formulation

We use the variational model proposed and analyzed in [1,2].

Let Ω be a bounded, open and connected subset of $\mathbb{R}^3$, representing the region occupied by the porous medium. Assume that the boundary of Ω is divided into two parts, Γ_1 and Γ_2. We suppose that Γ_1 and Γ_2 are Lipschitz two-dimensional manifolds. Let $T > 0$ be an arbitrary time instant; $Q := \Omega \times [0, T]$; $\Sigma_1 := \Gamma_1 \times [0, T]$; and $\Sigma_2 := \Gamma_2 \times [0, T]$.

Denote by s, p, and k the saturation, the pressure of the phase inside, and the hydraulic conductivity of the medium, respectively. Suppose that all processes occur in the range of validity of Darcy's law, which yields the following relationship between the flux q of the phase inside the porous medium, pressure, and hydraulic conductivity:

$$q = -k\nabla(p + \rho g z) \quad \text{in } Q,$$

where z is the vertical coordinate of the point $x \in \Omega$, g is the gravity acceleration, and ρ is the density of the phase. The mass conservation law, written in the form of the continuity equation for the water content inside any closed region of the medium, yields the following equation:

$$\dot{s} - \nabla\cdot(k\nabla(p + \rho g z)) = 0 \quad \text{in } Q.$$

Let P be a nonnegative function defined on Σ_2 to prescribe boundary data for the pressure p. Typically, P vanishes on those parts of Σ_2 which are in contact with air, and coincides with the corresponding hydrostatic pressure of the reservoir on parts of Σ_2 in contact with water.

On Σ_1, a homogeneous Neumann boundary condition is prescribed as

$$k\nabla(p + \rho g z)\cdot\vec{n} = 0,$$

where $\vec{n}$ is the outward normal unit vector to Ω. This condition means that there is no flux through Σ_1.

On Σ_2, the following condition holds:

$$k\nabla(p + \rho g z)\cdot\vec{n}\,(p - q) \le 0 \qquad \forall\, q: \Sigma_2 \to \mathbb{R} \text{ s. t. } q^+ = P. \tag{1}$$

This variational inequality is often referred to as Signorini

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boundary condition. For an interpretation and derivation of (1), see e.g. [1,2].

3. A Preisach model for capillary hysteresis

The (phenomenological) Preisach model arises in the following manner (see e.g [6,7],). Consider a single small pore initially filled with the phase, i.e. $s=1$. If p decreases and reaches a critical value b, the phase is instantly sucked out from the pore so that s jumps to 0. If p increases, nothing happens until p reaches some point $a \geq b$, and then the phase occupies the pore again.

Mathematically speaking, this phenomenon is described by the so-called relay operator. Following [3] we define this operator as follows: Let

$$v := \frac{a+b}{2}, \quad r := \frac{a-b}{2} \quad \text{and} \quad s_0 \in \{0, 1\}.$$

For $t \in [0, T]$, set

$$\mathfrak{S}(t) := \{\tau \in [0, t];\ |p(\tau) - v| = r\}$$

and $\tau_t := \max \mathfrak{S}(t)$, if $\mathfrak{S}(t) \neq \emptyset$. The relay operator is defined by the following relations:

$$\mathcal{R}_{v,r}[s_0, p](0) := \begin{cases} +1 & \textbf{if } p(0) \geq v + r, \\ 0 & \textbf{if } p(0) \leq v - r, \\ s_0 & \textbf{if } p(0) \in [v - r, v + r] \end{cases}$$

and

$$\mathcal{R}_{v,r}[s_0, p](t) := \begin{cases} \mathcal{R}_{v,r}[s_0, p](0) & \textbf{if } \mathfrak{S}(t) = \emptyset, \\ \dfrac{1}{r}(p(\tau_t) - v) & \textbf{if } \mathfrak{S}(t) \neq \emptyset. \end{cases}$$

Thus, the behavior of a single pore is described by the parameters v and r. It is reasonable to consider the medium as an ensemble of pores whose parameters v and r are distributed according to a nonnegative measure $\mu(v, r)$ defined on $\mathbb{R}^2_+$. The saturation s is then related to p by the Preisach operator in the following way:

$$s(t) := \int_{\mathbb{R}^2_+} \mathcal{R}_{v,r}[s_0(v, r), p](t)\, d\mu(v, r). \tag{2}$$

4. Mathematical analysis of the model

According to [1,2], the weak formulation of the physical model described above reads as follows.

Let $P \in L^2(0, T; H^1(\Omega))$ be such that $\gamma_0 P \geq 0$ a.e. on Σ_2, where γ_0 is the trace operator. Let $a_0 > 0$, $\mathcal{W}$ be the Preisach operator defined by (2), and set

$$K := \{q \in L^2(0, T; H^1(\Omega));\ \gamma_0 q^+ = \gamma_0 P \text{ a. e. on } \Sigma_2\}.$$

We consider the following problem.

Problem 1. Given an initial saturation s_0, find $p \in L^2(0, T; H^1(\Omega)) \cap L^2(\Omega; C([0, T]))$ and $s \in H^1(0, T; L^2(\Omega))$ such that

$$s(x, 0) = s_0(x) \quad \text{for a. a. } x \in \Omega;$$

$$s(x, t) := a_0 p(x, t) + \mathcal{W}[p(x, \cdot), s_0(x)](t) \quad \forall\, t \in [0, T] \text{ and a. a. } x \in \Omega;$$

$$\iint_Q [\dot{s}(p - q) + k(\mathcal{W}[p])\nabla(p + gz)\nabla(p - q)]\, dx\, dt \leq 0 \qquad \forall\, q \in K.$$

4.1. The existence result

The following result is obtained in El Behi-Gornostaeva [8]. We impose the following assumptions on the problem data.

Assumption 1 (*Preisach density*). Assume that

1. There exists a density function $\psi(v, r)$, such that the antisymmetric part of ψ belongs to $L^1(\mathbb{R}^2_+)$, i.e.
$$\psi_a(v, r) := \tfrac{1}{2}(\psi(v, r) - \psi(-v, r)) \in L^1(\mathbb{R}^2_+),$$
and
$$d\mu(v, r) = \psi(v, r)\, dv\, dr,$$
holds.
2. There is a function $\beta_1 \in L^1_{loc}(0, \infty)$ such that
$$0 \leq \psi(v, r) \leq \beta_1(r) \quad \text{for. a. a. } (v, r) \in \mathbb{R}^2_+.$$
3. It holds $\partial\psi/\partial v \in L^\infty_{loc}(\mathbb{R}^2_+)$.

Assumption 2 (*Initial and boundary data*). Assume that

1. k is a nondecreasing and Lipschitz function of the saturation s.
2. $0 < \underline{s} \leq s_0 < \bar{s} < 1$ a.e. in Ω.
3. There is $\alpha \in (0, 1)$ such that
$$p_0, s_0 \in H^1(\Omega) \cap C^\alpha(\overline{\Omega}) \cap L^\infty(\Omega).$$
4. The function P has the following regularity:
$$P \in W^{1,\infty}(0, T; L^\infty(\Omega)) \cap H^2(0, T; L^2(\Omega)) \cap H^1(0, T; H^1(\Omega)),$$
$$\gamma_0 P \in L^\infty(\Sigma_2).$$
5. There is $\beta \in (0, 1)$ such that
$$P \in C^{\beta,\beta/4}(Q \cup \Sigma_2).$$
6. The set $\{x \in \Gamma_2:\ P(x, t) > 0\}$ does not depend on the time t.

Proposition 1. *If Assumptions 1 and 2 hold, there exists a solution pair (s,p) to Problem 1.*

4.2. Sketch of the proof

Proposition 1 can be proven using the Rothe scheme. First, construct an implicit time discretization of Problem 1 as follows. Choose $m \in \mathbb{N}$, set $h := T/m$, and define

$$K^n_m := \{q \in H^1(\Omega);\ \gamma_0 q^+ = \gamma_0 P(\cdot, nh) \text{ a. e. on } \Gamma_2\}.$$

For every $1 \leq n \leq m$ find p^n_m and s^n_m satisfying the relations

$$p^n_m \in K^n_m;$$

$$s^n_m(x) := a_0 p^n_m(x) + \mathcal{W}^n_m[p^n_m(x), s^{n-1}_m(x)] \quad \text{a. e. in } \Omega;$$

$$\int_\Omega \frac{s^n_m - s^{n-1}_m}{h}(p^n_m - q)\, dx + \int_\Omega k(\mathcal{W}^{n-1}_m[p^{n-1}_m])\nabla(p^n_m + gz)\nabla(p^n_m - q)\, dx \leq 0 \tag{3}$$

for all $q \in K^n_m$. The unique solvability of this approximate problem follows from the Lax–Milgram lemma.

Second, using the De Giorgi iteration technique, derive the estimate

$$\| p^n_m \|_{L^\infty(\Omega)} < C,$$

where the constant C is independent of m and n. Application of the same technique yields the existence of some α, $0 < \alpha < 1$, such that